Figures

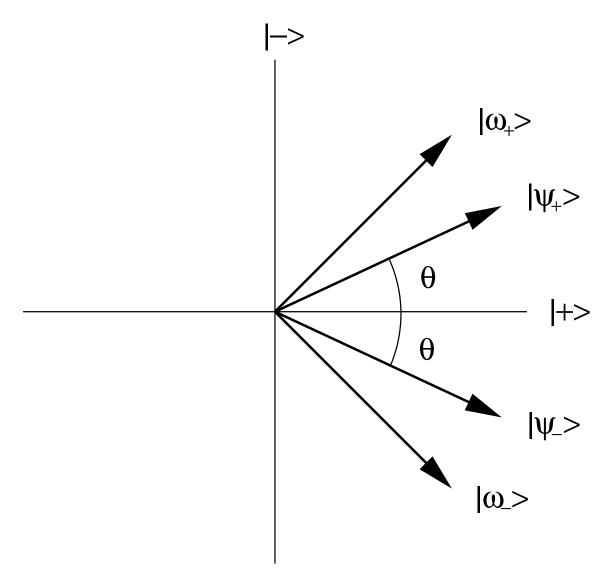


FIG. 1. States involved in the Helstrom measurement. The nonorthogonal states to be distinguished are $|\psi_{\pm}\rangle$. If these states are real-valued, then they can be represented in a plane. The angle can also be taken to be acute: if the angle is obtuse, then we can rotate one of the states by π radians without altering its physical meaning, and the new pair of states will subtend an acute angle. Any such pair of states can be written as $|\psi_{\pm}\rangle = \cos\theta|+\rangle \pm \sin\theta|-\rangle$, where $|\pm\rangle$ are some pair of orthogonal states. If the states $|\psi_{\pm}\rangle$ have equal a priori probabilities, then the optimal measurement is a von Neumann measurement in the basis $|\omega_{\pm}\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$, which gives the minimum error probability.

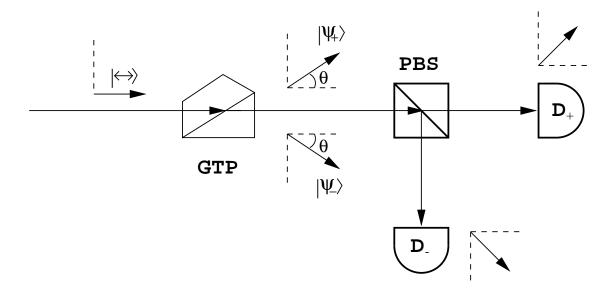


FIG. 2. Depiction of the apparatus used in the Barnett-Riis experiment. Weak optical pulses (\sim 0.1 photons/pulse) were prepared in a state of horizontal polarisation $|\leftrightarrow\rangle$. These then encountered a Glan-Thompson polariser. This is a polarisation filter, whose transmission axis was set to be in either the direction of $|\psi_{+}\rangle$ or $|\psi_{-}\rangle$. The pulse either emerged in one of these states, or was absorbed. The output pulses then encountered a polarising beamsplitter PBS. This was oriented at $\pi/4$ radians to the horizontal, so as to transmit the component in the direction $(|\leftrightarrow\rangle + |\downarrow\rangle)$ to detector D_{+} , and reflect the component in the orthogonal direction $(|\leftrightarrow\rangle - |\downarrow\rangle)$, towards the detector D_{-} . If a detection at D_{+} is taken to indicate that the initial state was $|\psi_{+}\rangle$, and likewise with D_{-} and $|\psi_{-}\rangle$, then this measurement gives the minimum probability of error.

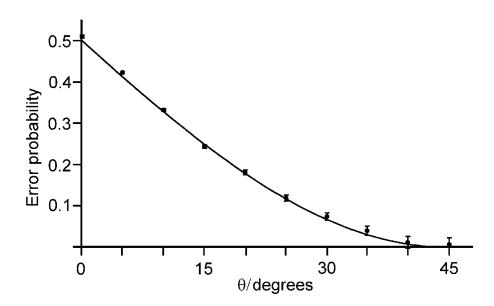


FIG. 3. Results of the Barnett-Riis experiment to discriminate between two optical polarisation states, $|\psi_{\pm}\rangle = \cos\theta| + \rangle \pm \sin\theta| - \rangle$, with minimum error probability. The error probability is shown versus the angle θ . Experimental data for specific values of θ are shown in comparison with the continuous Helstrom limit in Eq. (2.10). Clearly the agreement between theory and experiment is excellent.

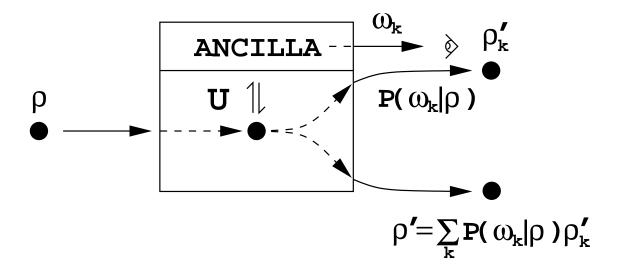


FIG. 4. Diagram illustrating the general structure of a quantum operation. A quantum system is initially prepared in some state, represented by the density operator ρ . It is then coupled to an ancilla system. The pair of systems evolves for some time according to some unitary operator U, following which a von Neumann measurement may be performed on the ancilla. The final state if the original system depends upon whether or not the measurement result is recorded. If it is, and result ω_k is obtained, then the state of the original system undergoes the transformation $\rho \rightarrow \rho_k' = A_k \rho A_k^{\dagger}/P(\omega_k|\rho)$, where the A_k are linear operators satisfying $\sum_k A_k^{\dagger} A_k = 1$. Also, $P(\omega_k|\rho)$ is the probability of obtaining the result ω_k , and is given by ${\rm Tr} \rho A_k^{\dagger} A_k$. If, on the other hand, the measurement result is not recorded, or no measurement is performed on the ancilla, then the final state of the system is simply the sum of all ρ_k' weighted by their respective probabilities, that is, $\rho \rightarrow \rho' = \sum_k P(\omega_k|\rho)\rho_k'$.

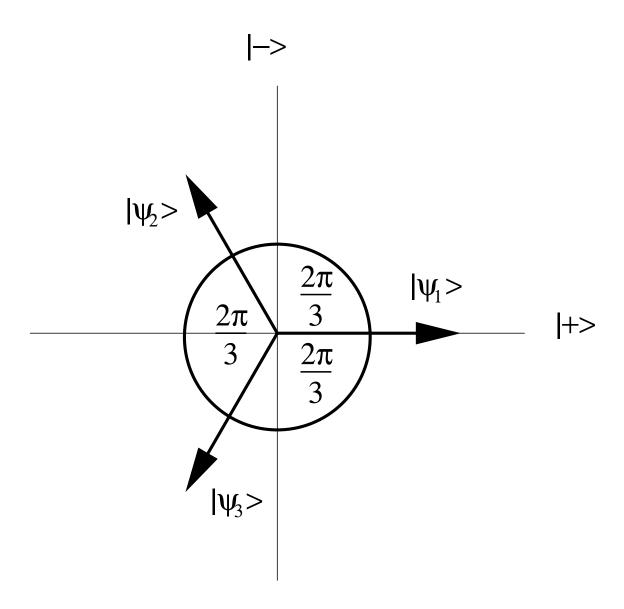


FIG. 5. Illustration of states forming the 'trine' ensemble. The states $|\psi_j\rangle$, where j=1,2,3, are linear combinations of a pair of orthogonal states $|\pm\rangle$. They have a real representation in this basis, which enables us to represent them in a plane. Each state is obtained from its predecessor by a unitary transformation, this being a counterclockwise rotation of $2\pi/3$ radians. Three consecutive such rotations are equivalent to the identity operator. As such, applying the rotation to the final state $|\psi_3\rangle$ gives the initial state $|\psi_1\rangle$.

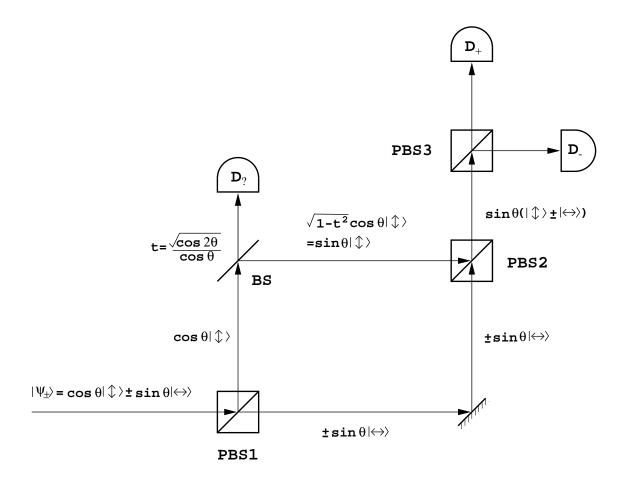


FIG. 6. Optical measurement to unambiguously discriminate between the non-orthogonal photon polarisation states $|\psi_{\pm}\rangle = \cos\theta |\uparrow\rangle \pm \sin\theta |\leftrightarrow\rangle$ at the Ivanovic-Dieks-Peres bound. The photon enters the apparatus at the bottom left, where a polarising beamsplitter PBS1 transmits the horizontal component and reflects the vertical one. The horizontal component propagates counter-clockwise to a second polarising beamsplitter PBS2. Meanwhile, the vertical component encounters an ordinary beamsplitter BS with transmission coefficient t given by Eq. (3.7). If the photon is transmitted, it will register at detector D_7 . Both states $|\psi_{\pm}\rangle$ have equal probability of giving rise to a detection here, so such outcomes give no information about the state. These are inconclusive results. If, on the other hand, this component is reflected, its amplitude is multiplied by the reflection coefficient $r = \sqrt{1-t^2}$, which results in both the vertical and horizontal components of both the states being equal. When the horizontal and vertical components are recombined by polarising beamsplitter PBS2, the resulting states, corresponding to whether the initial state was $|\psi_{+}\rangle$ or $|\psi_{-}\rangle$. These states can thus be unambiguously discriminated by a final, suitably oriented polarising beamsplitter PBS3.

Quantum State Discrimination

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There are fundamental limits to the accuracy with which one can determine the state of a quantum system. I give an overview of the main approaches to quantum state discrimination. Several strategies exist. In quantum hypothesis testing, a quantum system is prepared in a member of a known, finite set of states, and the aim is to guess which one with the minimum probability of error. Error free discrimination is also sometimes possible, if we allow for the possibility of obtaining inconclusive results. If no prior information about the state is provided, then it is impractical to try to determine it exactly, and it must be estimated instead. In addition to reviewing these various strategies, I describe connections between state discrimination, the manipulation of quantum entanglement, and quantum cloning. Recent experimental work is also discussed.

I. INTRODUCTION

The information which describes the state of a physical system is that which is required to predict its future evolution and its effect on other systems. Among the other physical systems whose dynamics it can influence are measuring devices. The state then contains all information that can be extracted by measurement and thus, all information that we can acquire about the system.

The state of a classical system is described by dynamical variables. For a one dimensional point particle, these are its position q and momentum p. If we have complete knowledge of the values of these variables, and also of the equations that describe their temporal evolution, then we can predict, with arbitrarily high accuracy, the future state of the system, and how it will interact with other systems.

The assumption that we can measure these variables with arbitrarily high accuracy is, however, an idealisation. In addition to the fact that the results of any real experiment will be affected by some uncontrollable noise, our measuring devices have finite precision and can only record a finite amount of information. This implies that they will not be able to record exactly the values of continuous variables which form the basis of the descriptions of most classical systems. Often then, we must settle for an approximate description of the state. In our simple example of a one dimensional point particle, practical limitations on our ability to measure q and p precisely might lead us to use a joint probability density $\varrho(q,p)$ instead. Often, such a description is adequate. However, there do exist situations where even the smallest, finite

uncertainty in our knowledge of the state variables will be amplified over time to such large proportions that long term prediction of the behaviour of the system becomes impossible. As is well-known, this extreme sensitivity to initial conditions is one of the chief hallmarks of the phenomenon of dynamical chaos [1].

As a matter of principle though, there are no fundamental limitations on the precision with which we can determine the state variables of a classical system. This is because the state variables are also observables. The amount of information that we can acquire about observables is always increasing as more refined measurements become possible. This equivalence of state information and the information that is potentially accessible through measurement is a highly non-trivial feature of classical physics which, despite its transparent nature, should not be taken for granted. Classical physics is only an approximate description of our world, and currently, our most accurate description of it is provided by quantum theory. Here, states and observables are completely distinct concepts.

The state of a quantum system is represented by a normalised vector $|\psi\rangle$ in a complex, linear vector space. The observable properties of quantum systems, by contrast, are the same as those of classical systems: position, momentum etc. Unlike their classical counterparts, they are not represented by simple numerical variables which evolve deterministically over time. Instead, they are represented by Hermitian operators on the vector space. We shall make no distinction in what follows between observables and their corresponding operators, and shall denote both aspects of a generic observable by Ω .

Knowledge of the state vector is instrumental in making predictions about the outcome of measuring an observable Ω . As is discussed in every introductory quantum mechanics text, e.g. [2], these predictions are not generally of a deterministic nature, and are concerned instead with statistical quantities. The average, or expectation value of Ω for an ensemble of quantum systems all prepared with the same state vector $|\psi\rangle$ is given by the inner product of $|\psi\rangle$ and $\Omega|\psi\rangle$,

$$\langle \Omega \rangle = \langle \psi | \Omega | \psi \rangle. \tag{1.1}$$

In order to treat a single system, we express Ω in terms of its eigenvalues ω_k and orthogonal eigenstates $|\omega_k\rangle$:

$$\Omega = \sum_{k} \omega_k |\omega_k\rangle \langle \omega_k|. \tag{1.2}$$

The eigenvalues ω_k are the values that the observable Ω can take. Inserting Eq. (1.2) into Eq. (1.1), we obtain

$$\langle \Omega \rangle = \sum_{k} \omega_k |\langle \psi | \omega_k \rangle|^2.$$
 (1.3)

The expectation value of any quantity is equal to the sum of the values that it can assume, multiplied by their respective probabilities. From this, we infer that the probability $P(\omega_k|\psi)$ of obtaining the result ω_k for the state $|\psi\rangle$ is

$$P(\omega_k|\psi) = |\langle\psi|\omega_k\rangle|^2. \tag{1.4}$$

If the result ω_k is obtained, then the state of the system following the measurement will become $|\omega_k\rangle$. This is the contentious process of state vector reduction, or 'collapse of the wavefunction'. Such a measurement is known as a von Neumann measurement.

The vector $|\psi\rangle$ is the most complete specification of the state of a quantum system. As is the case in classical physics, it is not always possible to specify the state exactly. By analogy with the probability density $\varrho(q,p)$, in dealing with quantum systems we will sometimes have to use a probability distribution for the state vector itself. This is described by a density operator,

$$\rho = \sum_{r} p_r |\psi_r\rangle\langle\psi_r|. \tag{1.5}$$

Here, p_r is the probability that the state vector of the system is $|\psi_r\rangle$, and $\sum_r p_r = 1$. When a quantum system is known to have a specific state vector, it is said to be in a *pure state*. Otherwise, it is in a *mixed state*. The generalisation of Eq. (1.4) for the probability of obtaining result ω_k when the system is in a possibly mixed state ρ is

$$P(\omega_k|\rho) = \langle \omega_k|\rho|\omega_k \rangle = \text{Tr}\rho|\omega_k \rangle \langle \omega_k|. \tag{1.6}$$

Here, 'Tr' stands for the trace operation. The trace of an operator is, in a matrix representation, the sum of its diagonal elements. The numerical value of this sum is basis independent. So, if we choose the basis to be that formed by the eigenstates $|\omega_k\rangle$ of Ω , then the trace is seen to be the sum of the corresponding eigenvalues ω_k . One can quite easily show, using Eq. (1.5), that $\text{Tr}\rho = 1$. From Eq. (1.6), it follows that the expectation value of Ω is

$$\langle \Omega \rangle = \sum_{k} \omega_k \langle \omega_k | \rho | \omega_k \rangle = \text{Tr} \rho \Omega.$$
 (1.7)

The evident distinction between states and observables in quantum mechanics begs the question: to what extent can we determine the state of a quantum system? The ability to do this would confer many benefits. The most obvious of these is that we would be able to evaluate the probability distribution for the results of any measurement that we might wish to perform upon a quantum system about whose state we have no prior information. Another less obvious benefit, which we shall later see, is that it would allow two parties to communicate across

arbitrarily large distances, instantaneously, in violation of the special theory of relativity.

The state itself is not an observable in quantum mechanics. As it happens, the impossibility of measuring the state has benefits of its own. As was demonstrated initially by Bennett and Brassard [3], the impossibility of measuring $|\psi\rangle$ precisely permits the existence of provably secure protocols for the transmission of confidential information. The security of quantum cryptographic protocols, unlike that of classical ones, is a consequence of physical theory [4,5]. The security of classical protocols depends upon unproven assumptions about the complexity of the decoding problem. One of the most widely used cryptosystems, the RSA cryptosystem, exploits the difficulty of the problem of reducing a large number to its set of prime factors [6]. No efficient classical algorithm for carrying out this task has been discovered. However, it has not been proven that one does not exist. In fact, it has recently been shown by Shor that an efficient algorithm does exist for quantum computers [7,8].

In this article, we will examine the problem of determining the state of a quantum system. Although the state is not, strictly speaking, an observable, through a judicious choice of legitimate observables, we can obtain information about it. Several strategies for state discrimination exist. The one we would use in any particular situation depends upon the type of information about the state we wish to obtain, and also on any prior information that we might possess.

The first strategy we examine is quantum hypothesis testing. Here, we are given a system whose state belongs to a known, finite set. Our aim is to guess, with the minimum probability of error, which of these states the system is in.

In the course of our discussion, we shall encounter the elegant formalism of generalised quantum measurements. This is based on a few necessary and sufficient conditions which any physically possible operation on a quantum system must satisfy.

Sometimes errors can be avoided altogether if we allow for the possibility of inconclusive results. We will see how this can be achieved. Another scenario is when the state does not belong to some finite, known set, and can be any state in the entire vector space. Under these circumstances, the set of possible states is infinite. Since any measurement can record only a finite amount of information, it is necessary to consider only a finite, but suitably large set of possible states. These states are known as guess states. An appropriate measurement strategy is one which selects the guess state which the actual state most closely resembles, as often as possible. This is known as quantum state estimation.

We shall also explore the relationship between trying to discriminate between quantum states and other matters, such as the problem of cloning quantum states, and the manipulation of nonlocal correlations between quantum systems, which can exist due to the strange phenomenon of quantum entanglement.

II. QUANTUM HYPOTHESIS TESTING

A. Basic strategy

In this article, we shall see that several approaches to the problem of quantum state discrimination exist. All of them refer to the same basic scenario. One party, conventionally called Alice, prepares a quantum system in a member of a set of quantum states. She might not prepare all of these states with the same probability. She then passes the system onto her colleague, Bob. His task is obtain as much information about the state which she prepared as he possibly can. Here, we use the term 'information' in a broad sense, and do not refer as yet to any specific measure. The main differences between the state discrimination strategies considered in this article correspond to the different types of information that Bob might wish to obtain.

In this section, we will assume that some information about the state is given to Bob. He is told what the possible states of the system are, and he is also told the probability that the system was prepared in each of them. Here, we will consider only the situation where there are N possible states, represented by density operators ρ_j , where $j=1,\ldots,N$, for some finite N. Bob is also told the probability, η_j , that the system was prepared in each of them. These probabilities, known as the a priori probabilities, satisfy

$$\sum_{j=1}^{N} \eta_j = 1, \tag{2.1}$$

since the system will, with certainty, be prepared in one of the states ρ_j .

Historically, the first strategy for state discrimination was that advanced by Helstrom [9]. This strategy is known as quantum hypothesis testing. In his attempt to determine the state of the system, Bob performs some measurement. The key feature of quantum hypothesis testing, as opposed to some other strategies, in particular, the one we shall examine in the next section, is that after his measurement, he is required, on the basis of his experimental results, to make a decision as to what the state was. He is not allowed to say 'don't know'. We will see that, if the states are not orthogonal, then no test exists which allows him to guess correctly all of the time, so that there will, in general, be a non-zero probability of error, which we shall denote by P_E . Likewise, we will write the probability of correctly determining the state as $P_D = 1 - P_E$.

Since there are N states, his experiment must have N outcomes, which we call ω_k . Following this kind of test, if Bob obtains the result ω_j , he makes the hypothesis that the state given to him by Alice was ρ_j .

To determine the probability of error, Bob needs to know the a priori probability η_j of being given the state ρ_j and the probability, given that ρ_j was sent, that he

will obtain the result ω_k , for all j, k. The probabilities form the *channel matrix* $[P(\omega_k|\rho_j)]$. The elements of this matrix satisfy the *completeness* condition

$$\sum_{k=1}^{N} P(\omega_k | \rho_j) = 1. \tag{2.2}$$

This expresses the fact that, no matter which ρ_j Bob receives, his measurement will, with certainty, yield one of the outcomes ω_k . The total error probability P_E is found to be

$$P_E = 1 - P_D = 1 - \sum_{j=1}^{N} \eta_j P(\omega_j | \rho_j).$$
 (2.3)

Quantum hypothesis testing actually belongs to a more general class of strategies known as quantum Bayes' strategies [9]. The general quantum Bayes strategy assigns a cost C_{kj} to making hypothesis ω_k when the state was ρ_j . The coefficients C_{kj} are known as the elements of the Bayes' cost matrix. The scenario can easily be understood in terms of gambling. Alice sends Bob one of the states ρ_j . If Bob says ' ω_k ', then he must pay Alice C_{kj} currency units. Some elements of the cost matrix can be negative, in which case Alice pays Bob, enabling him to win money. The average amount that Bob will pay Alice is then given by the Bayes' cost function

$$C_B = \sum_{jk} \eta_j C_{kj} P(\omega_k | \rho_j). \tag{2.4}$$

For a fixed cost matrix $[C_{kj}]$ and a priori probabilities η_j , Bob's task is to minimise the overall Bayes' cost C_B , that is, to use a measurement which minimises his average payout to Alice. The only quantities that Bob is free to vary are the channel matrix elements $P(\omega_k|\rho_j)$. Since the possible states ρ_j are fixed, the only thing that Bob is free to vary is his measurement strategy.

The form of the cost matrix depends on the particular situation. In general, some errors may be more costly than others. If every error has equal cost, then the corresponding Bayes' cost is closely related to the average error probability. There is no cost when the result is correct, so the diagonal elements of the cost matrix, C_{jj} , are zero. Let all other elements have an associated cost c: that is, $C_{jk} = c$ for $j \neq k$. Then one can show, using the definitions of the error probability and Bayes' cost in Eqs. (2.3) and (2.4), and the completeness condition in Eq. (2.2), that the Bayes' cost and error probability are related by $C_B = cP_E$. When all errors have the same cost c, minimisation of the Bayes' cost is equivalent to minimisation of the error probability.

The lowest value of P_E is obtained by varying the elements of the channel matrix, $P(\omega_k|\rho_j)$. As a consequence of the non-trivial nature of the measurement process in quantum mechanics, the form of this matrix cannot be specified arbitrarily. Consider for example a von Neumann measurement of an observable Ω . Let us assume

that the states ρ_j are pure states $|\psi_j\rangle\langle\psi_j|$. We would like to associate each outcome with a unique state, so that if Bob obtains the result ω_j he will make the hypothesis that the state Alice sent him was $|\psi_j\rangle$. Clearly, the number N of possible states must be equal to the number of outcomes. Consequently, we must also then assume that the $|\psi_j\rangle$ are linearly independent. Otherwise, there would be more states than outcomes.

In the Introduction, we saw that each outcome of a von Neumann measurement corresponds to a different eigenvalue of an Hermitian operator Ω . The eigenvalues ω_j are the possible numerical values of Ω , considered as an observable property of the system. From Eq. (1.4), we see that the channel matrix elements are given by the square-overlaps between the $|\psi_j\rangle$ and the eigenstates of Ω :

$$P(\omega_k | \psi_j) = |\langle \psi_j | \omega_k \rangle|^2. \tag{2.5}$$

The diagonal elements of this matrix must all be equal to 1 if the error probability is to vanish. This clearly gives the requirement that $|\psi_j\rangle = |\omega_j\rangle$, which cannot be the case if the $|\psi_j\rangle$ are non-orthogonal.

A simple von Neumann measurement of this kind, however, is often the most useful for the kind of strategy we are considering. In fact, it was proven in 1973 by Kennedy [10] that if we are attempting to distinguish between N pure states which are linearly independent, as we have been assuming, then there is always a von Neumann measurement which is optimal, in the sense that it can be used to obtain the smallest possible value of P_E . It follows that only orthogonal states can be perfectly discriminated.

B. Hypothesis testing for two pure states

The simplest set of linearly independent pure states, and historically the first set for which an explicit expression for the minimum error probability was obtained, is that of just two states. The problem of finding the minimum value of P_E for two pure states, which we shall simply simply by $|\psi_{\pm}\rangle$, was solved by Helstrom [9] and can be considered to be a pioneering work in quantum detection theory. Helstrom's optimal value of P_E is

$$P_E(\text{opt}) = \frac{1}{2} \left(1 - \sqrt{1 - 4\eta_+ \eta_- |\langle \psi_+ | \psi_- \rangle|^2} \right).$$
 (2.6)

Naturally, we would like to determine the von Neumann measurement which can be used to attain this limit. The corresponding basis states $|\psi_{\pm}\rangle$ have quite a simple form if we take the states $|\psi_{\pm}\rangle$ to be

$$|\psi_{\pm}\rangle = \cos\theta |+\rangle \pm \sin\theta |-\rangle \tag{2.7}$$

for some angle $0 \le \theta \le \pi/4$, and where $|\pm\rangle$ is an orthogonal basis for the space spanned by $|\psi_{\pm}\rangle$. For the states $|\psi_{\pm}\rangle$ in Eq. (2.7), the optimum detector states $|\omega_{\pm}\rangle$ are

$$|\omega_{\pm}\rangle = \frac{1}{\sqrt{2}} \left[\sqrt{1 \pm \xi} |+\rangle \pm \sqrt{1 \mp \xi} |-\rangle \right].$$
 (2.8)

Here, $\xi = \Delta \cos 2\theta / \sqrt{1 + \cos^2 2\theta (\Delta^2 - 1)}$ where $\Delta = \eta_+ - \eta_-$. For alternative expressions for the optimal detector states, see [11]. The optimum detection strategy for a pair of *mixed* quantum states has also been obtained. For a full discussion, see [12].

For two pure states with equal a priori probabilities $\eta_{+} = \eta_{-} = 1/2$, the optimum detector states in Eq. (2.8) have a much simpler form. and the optimum measurement has appealing geometrical properties. When the a priori probabilities are equal, we have $\Delta = 0$, which in turn implies that $\xi = 0$. The states $|\omega_{\pm}\rangle$ are then given simply by

$$|\omega_{\pm}\rangle = \frac{|+\rangle \pm |-\rangle}{\sqrt{2}} \tag{2.9}$$

and the minimum error probability simplifies to $P_E(\text{opt}) = [1 - (1 - |\langle \psi_+|\psi_-\rangle|^2)^{1/2}]/2$. All 4 states in Eqs. (2.7) and (2.9) are depicted in Figure 1. The symmetrical properties of the measurement states $|\omega_\pm\rangle$ with respect to the possible states $|\psi_\pm\rangle$ are clearly visible in the figure. The $|\omega_\pm\rangle$ are as close as they can be to the $|\psi_\pm\rangle$ whilst maintaining orthogonality. The reflection symmetry about the $|+\rangle$ -axis is due to the equality of the a priori probabilities η_\pm . We can also see from the figure that errors are unavoidable, since $|\omega_\pm\rangle$ is not orthogonal to $|\psi_\pm\rangle$.

The Helstrom measurement has recently been carried out in the laboratory by Barnett and Riis [13]. In this experiment, the two states were non-orthogonal photon polarisation states, having the form shown in Eq. (2.7), where the orthogonal states $|+\rangle$ and $|-\rangle$ were the horizontal $|\leftrightarrow\rangle$ and vertical $|\uparrow\rangle$ polarisation states respectively. The experimental arrangement used is shown in Figure 2. Pulses of light emerged from the left in the horizontally polarised state $|\leftrightarrow\rangle$. These were then heavily attenuated to the point where, on average, only 1 in 10 pulses contains a photon. This was done to make the probability of there being 2 or more photons per pulse negligible. A Glan-Thompson polariser GTP was then used to transform the photons into one of the states $|\psi_{\pm}\rangle$ in Eq. (2.7). The beam was then analysed at a polarising beam splitter PBS oriented at an angle of $\pi/4$ to the horizontal. To understand the action of this beamsplitter, we refer to the states $|\omega_{\pm}\rangle$ in Eq. (2.9), and again make the identifications $|+\rangle = |\leftrightarrow\rangle$ and $|-\rangle = |\uparrow\rangle$. A photon in the state $|\omega_{+}\rangle$ would be transmitted by the beam splitter, while $|\omega_{-}\rangle$ would be reflected. The transmitted and reflected states were fed to photodetectors D_{+} and D_{-} respectively. Correct results were obtained when a photon prepared in the state $|\psi_j\rangle$ was detected at D_j , where $j = \pm$. If the photon was detected at the other 'wrong' detector, an error ensued.

In the Barnett-Riis experiment, both states had equal a priori probabilities. The minimum error probability is then

$$P_E(\text{opt}) = \frac{1}{2}(1 - \sin 2\theta).$$
 (2.10)

Experimental results for the error probability for various values of θ are shown alongside the theoretical minimum in Figure 3.

Although the problem of finding the minimum error probability $P_E(\text{opt})$ for two states has been solved completely, it is generally difficult to find analytic expressions for more than two states. Standard von Neumann measurements of the kind which are optimal for two pure states, or indeed, any number of linearly independent states, cannot detect all of the states if they form a linearly dependent set. If the state space of the system is N dimensional, then a von Neumann measurement can have at most N outcomes. If there are more than N states, then some of these cannot be detected by the measurement.

Fortunately, the formalism of quantum mechanics does not restrict us to state transformations described by von Neumann measurements. This much is obvious from the transformation generated by the free evolution of a quantum system. This is described by the Schrödinger equation, and is quite unlike what happens during a measurement. In order to decide whether or not a given operation on the quantum state is physically realisable, it would be helpful to know what the general criteria are. These are firmly established, and form the basis for the elegant formalism of generalised quantum measurements [14], which we now describe.

C. Generalised measurements

Consider a quantum system initially prepared in the state ρ . An operation $\rho \rightarrow L(\rho)$ is carried out of the system. This operation has K distinguishable outcomes which, as before, we label ω_k , $k=1,\ldots,K$, with corresponding final density operators ρ'_k . In a von Neumann measurement, the probability of outcome ω_k is given by Eq. (1.6), that is, the trace of the product of the initial density operator and $|\omega_k\rangle\langle\omega_k|$. In the more general kind of measurement we describe here, the latter operators are replaced by more general operators, known as quantum detection operators, Π_k . By analogy with Eq. (1.6), the probability of obtaining result ω_k given the initial state ρ , is

$$P(\omega_k|\rho) = \text{Tr}\rho\Pi_k. \tag{2.11}$$

If ρ is a pure state $|\psi\rangle\langle\psi|$, then this probability $P(\omega_k|\psi)$ is simply

$$P(\omega_k|\psi) = \langle \psi | \Pi_k | \psi \rangle. \tag{2.12}$$

Naturally, $P(\omega_k|\psi)$ is always real. This implies that the quantum detection operators must be Hermitian. This probability must also be non-negative for all states.

Thus, the expectation value of Π_k must always be nonnegative. Operators whose expectation values are nonnegative for all possible states are said to be positive (semi-definite). They may be equivalently defined as operators whose eigenvalues are non-negative. One further constraint on the form of these operators comes from the requirement that the possible outcomes ω_k are exhaustive, which implies that $\sum_k P(\omega_k|\rho) = 1$ for all possible states. From this, it follows that the Π_k form a resolution of the identity,

$$\sum_{k} \Pi_k = 1. \tag{2.13}$$

The conditions we have just given are the necessary and sufficient conditions for the realisability of an experiment whose outcomes have the probability distribution $P(\omega_k|\rho)$ [14]. Such an operation is also commonly known as a *positive operator-valued measure* operation, or POVM, and the detection operators are called the elements of the POVM.

Often, and particularly in state discrimination, we are only interested in these probabilities, and not overly concerned about how the state of the system is transformed by the measurement. However, this is not always the case, indeed we shall be concerned about this issue in section IV. It is then useful to know what form this state transformation must take. To this end, consider the operator

$$A_k = U_k \Pi_k^{1/2}, (2.14)$$

where U_k is any unitary operator. From this expression, and from the fact that $U_k^{\dagger}U_k=1$, we can see that $\Pi_k=A_k^{\dagger}A_k$, and that the detection probability $P(\omega_k|\rho)$ can be alternatively expressed as ${\rm Tr}A_k^{\dagger}A_k\rho$. It can also be expressed as ${\rm Tr}A_k\rho A_k^{\dagger}$, since the trace of a product of operators is invariant under cyclic permutations. The post-measurement density operator, given that result ω_k was obtained, is

$$\rho_k' = \frac{A_k \rho A_k^{\dagger}}{P(\omega_k | \rho)}.$$
 (2.15)

The presence of the probability in the denominator serves to give $\text{Tr}\rho_k'=1$, normalising the state. If we do not actually record the result of the measurement, then the final density operator, which may simply be denoted by ρ' , is given by a distribution of the density operators ρ_k corresponding to the possible outcomes of the operation, weighted by their respective probabilities $P(\omega_k|\rho)$. That is

$$\rho' = \sum_{k} P(\omega_k | \rho) \rho_k' = \sum_{k} A_k \rho A_k^{\dagger}. \tag{2.16}$$

The formalism we have outlined appears to be more general than the description of quantum state changes given in introductory quantum mechanics texts. There, usually

only unitary operations and von Neumann measurements are discussed. It is easy to see that these operations are special cases of the generalised measurements we have just described. A von Neumann measurement of an operator Ω with the orthogonal eigenstates $|\omega_k\rangle$ can be expressed in terms of the operators $P_k = |\omega_k\rangle\langle\omega_k|$. These are projection operators, and are clearly Hermitian and positive, having eigenvalues 0 and 1. They also form a resolution of the identity

$$\sum_{k} P_k = 1, \tag{2.17}$$

which expresses the completeness of the orthogonal basis $|\omega_k\rangle$. They are also idempotent, that is, $P_k^2 = P_k$. It follows that the projectors P_k satisfy the properties required for them to be physical transformation operators A_k . Applying Eq. (2.15), we see that the possible postmeasurement states are just the eigenstates $|\omega_k\rangle$, in accordance with the idea of collapse of the wavefunction. If the measurement result is not recorded, substituting the P_k for A_k in Eq. (2.16) just gives a statistical mixture of the eigenstates, weighted by their respective probabilities $P(\omega_k|\rho)$.

At the other extreme, if only one A_k , which we may just call A, is non-zero, then the resolution of the identity in Eq. (2.13) implies that $A^{\dagger}A = 1$, i.e. that A is unitary. Equations (2.13) and (2.16) are both equivalent here, since there is only one 'outcome', which represents unitary evolution according to the von Neumann equation

$$i\hbar \frac{d\rho}{dt} = [H, \rho],$$
 (2.18)

where $[H,\rho]$ is the commutator $H\rho-\rho H$. This is simply the generalisation of the Schrödinger equation to cover mixed states. The solution is $\rho(t)=U\rho(0)U^{\dagger}$, where $U=\mathrm{e}^{\frac{-iHt}{\hbar}}$. Any unitary operator U can be written in this form for some Hamiltonian H. So, if we are sufficiently able to tailor the Hamiltonian H of our system, then we can generate any unitary evolution.

Despite its appearance, the generalised measurement formalism is not really more general than these two more familiar types of state transformation. An important result, known as the Naimark theorem [15], tells us that any generalised measurement can be realised with an ancillary system, a unitary operation and a von Neumann measurement. Specifically, if we wish to realise a generalised measurement with K outcomes, we need a large ancillary system. The system of interest is then made to interact unitarily with the ancilla. In general, this results in the original system and the ancilla becoming entangled. Entanglement is a feature of quantum mechanics we shall examine at greater length in section IV. Following this interaction, a von Neumann measurement is performed on the ancilla. As a consequence of this entanglement, this measurement also transforms the state of our original system, and the results of this measurement give rise to the corresponding transformations in Eq. (2.15). The effect of a generalised measurement, implemented by a unitary interaction with an ancilla, followed by a measurement on the latter, is illustrated in Figure 5.

D. Hypothesis testing for multiple states

Returning now to the problem of state discrimination, Bob has in his possession a quantum system prepared in one of the N states ρ_j , with a priori probabilities η_j . His aim is to determine the strategy with N outcomes whose detection operators Π_k give the minimum value of the error probability. If outcome ω_j is taken to correspond to detection of the state ρ_j , then the minimum value of the error probability is obtained from the fact that the probability of correctly identifying the state ρ_j will be $\mathrm{Tr} \rho_j \Pi_j$. The sum of these probabilities for the N states ρ_j , weighted by their a priori probabilities η_j , gives the total probability P_D that the state will be correctly identified. The error probability P_E is equal to $1-P_D$, giving

$$P_E = 1 - \sum_{j} \eta_j \text{Tr} \Pi_j \rho_j. \tag{2.19}$$

Holevo [16] and Yuen *et al* [17] independently determined the necessary and sufficient conditions that a set of detection operators must satisfy to give the minimum value of P_E . These are

$$\Pi_j [\eta_j \rho_j - \eta_k \rho_k] \Pi_k = 0, \qquad (2.20)$$

$$\Gamma - \eta_i \rho_i \ge 0, \tag{2.21}$$

where we have defined an operator Γ known as the *Lagrange operator*

$$\Gamma = \sum_{k} \eta_k \Pi_k \rho_k, \qquad (2.22)$$

which, as a consequence of the condition in Eq. (2.20), is Hermitian (this can be seen by summing Eq. (2.20) over both j and k, and making use of the resolution of identity in Eq. (2.13).) One important kind of ensemble of states for which the optimum strategy can be derived analytically are pure states with equal a priori probabilities, $\eta_j = 1/N$, which are also symmetric [9,18]. A set of states is symmetric if it satisfies the following conditions:

$$|\psi_j\rangle = U|\psi_{j-1}\rangle = U^{j-1}|\psi_1\rangle,$$
 (2.23)

$$U|\psi_N\rangle = |\psi_1\rangle,\tag{2.24}$$

for some unitary operator U. We see that U transforms each state into its successor, and the final state back to the initial state. The optimum measurement for these states is the so-called square-root measurement. We define the operator:

$$\Phi = \sum_{j=1}^{N} |\psi_j\rangle\langle\psi_j|. \tag{2.25}$$

The optimum detection operators Π_j are of the form

$$\Pi_j = |\omega_j\rangle\langle\omega_j|,\tag{2.26}$$

where the, in general unnormalised, and non-orthogonal states $|\omega_i\rangle$ are given by

$$|\omega_j\rangle = \Phi^{-1/2}|\psi_j\rangle. \tag{2.27}$$

It is because of the presence of $\Phi^{-1/2}$ on the right hand side of this expression that this measurement is known as the square-root measurement. For equally-probable symmetric states, this measurement attains the minimum error probability

$$P_E(\text{opt}) = 1 - \frac{1}{N} \sum_{j=1}^{N} |\langle \psi_j | \Phi^{-1/2} | \psi_j \rangle|^2.$$
 (2.28)

The simplest set of symmetric states is that of just two states, which we examined above. Looking back at Eq. (2.7), we see that if the orthogonal basis states $|\pm\rangle$ are to be regarded as the spin up/down states of a spin-1/2 particle with respect to the z-axis, then the z component of the spin vector, σ_z , acts on these states to give $\sigma_z|\pm\rangle=\mp|\mp\rangle$. Using this property, it can easily be seen that $\sigma_z|\psi_\pm\rangle=|\psi_\mp\rangle$. This operator is unitary and satisfies $\sigma_z^2=1$, so these states satisfy the symmetric states conditions in Eqs. (2.23-2.24).

The next simplest case is that of three states. Three symmetric photon polarisation states are

$$|\psi_1\rangle = |\leftrightarrow\rangle,\tag{2.29}$$

$$|\psi_2\rangle = \frac{-|\leftrightarrow\rangle + \sqrt{3}|\uparrow\rangle}{2},$$
 (2.30)

$$|\psi_3\rangle = \frac{-(|\leftrightarrow\rangle + \sqrt{3}|\uparrow\rangle)}{2}.$$
 (2.31)

These states are illustrated schematically in Figure 5, where we see that they are distributed around a circle, with equal angular spacing of $2\pi/3$ radians. This ensemble of states is sometimes called the *trine* ensemble. If these states have equal a priori probabilities, then the minimum error probability is equal to 1/3, and the optimum strategy is given by the detection operators

$$\Pi_j = \frac{2}{3} |\psi_j\rangle\langle\psi_j|. \tag{2.32}$$

The first method of carrying out the optimum state discrimination measurement for a trine ensemble of photon polarisation states was proposed by Sasaki *et al* [19]. Phoenix *et al* [20] explored the potential of this ensemble of states in quantum cryptography, showing that, for a certain, novel three state key distribution protocol, it is

the trine ensemble which can be used to generate secret key bits most efficiently.

The square-root measurement for three symmetrical photon polarisation states can be performed with current technology. Indeed, at the time of writing, experimental demonstrations of both this measurement, and also the more complex minimum error probability discrimination of four non-coplanar states arranged as a tetrahedron, have just been carried out by Clarke *et al* [21].

III. UNAMBIGUOUS STATE DISCRIMINATION

A. Error-free discrimination between two non-orthogonal states.

As we have seen, the formalism of generalised measurements offers greater scope for the possibility of discriminating between non-orthogonal quantum states than simple von Neumann measurements. One of the main advantages conferred by generalised measurements is the fact that the number of distinguishable outcomes can be arbitrarily large. The number of outcomes possible with a von Neumann measurement is restricted to be no greater than the number of dimensions of the system's state space. This means that quantum hypothesis testing, with a full set of outcomes corresponding to each of the states, is not generally possible with von Neumann measurements, in particular, if the states are linearly dependent. As we saw in the case of the trine ensemble, it is necessary under such circumstances to use a generalised measurement.

Being able to perform measurements with an arbitrary number of outcomes, we might ask ourselves if there is anything to be gained if not every outcome need correspond to the detection one of the states. In other words, do we gain anything if we allow our measurement to have inconclusive results? This issue was first examined in 1987 by Ivanovic [22], who made the startling discovery that the possibility of occasionally obtaining inconclusive results permits error free discrimination between non-orthogonal states. Ivanovic showed that when the result of this measurement is not inconclusive, it is always correct.

To see how this may be done, consider again the two states $|\psi_{\pm}\rangle$ defined in Eq. (2.7). Let us now introduce the additional states

$$|\psi_{\pm}^{\perp}\rangle = \sin\theta |+\rangle \pm \cos\theta |+\rangle.$$
 (3.1)

Notice that $|\psi_{+}^{\perp}\rangle$ is orthogonal to $|\psi_{-}\rangle$, and likewise with $|\psi_{-}^{\perp}\rangle$ and $|\psi_{+}\rangle$. Consider now a generalised measurement described by the following detection operators:

$$\Pi_{\pm} = \frac{P_{\pm}}{|\langle \psi_{\pm}^{\perp} | \psi_{\pm} \rangle|^2} |\psi_{\pm}^{\perp} \rangle \langle \psi_{\pm}^{\perp} |, \qquad \Pi_{?} = 1 - \Pi_{+} - \Pi_{-}.$$
(3.2)

The meaning of the coefficients P_{\pm} will become apparent shortly. Since $|\psi_{+}^{\perp}\rangle$ is orthogonal to $|\psi_{-}\rangle$, we see that $\langle \psi_{-}|\Pi_{+}|\psi_{-}\rangle = 0$, and so the probability of obtaining the result '+' for this state must be zero. Likewise, $\langle \psi_{+}|\Pi_{-}|\psi_{+}\rangle = 0$, so we will never obtain the result '-' for the initial state $|\psi_{+}\rangle$. Thus, whenever we obtain one of these two results, we can retrodict *exactly* what the initial state was. We can easily see that

$$\langle \psi_{\pm} | \Pi_{\pm} | \psi_{\pm} \rangle = P_{\pm}. \tag{3.3}$$

This means that P_{\pm} is the probability, given that the system was prepared in the state $|\psi_{\pm}\rangle$, that this state will be identified unambiguously.

Unless the states are orthogonal, these probabilities cannot attain the value of 1. There is a third result, the inconclusive result '?', the probability of which for each state is equal to the expectation value of the operator $\Pi_{?}$. As with the quantum hypothesis testing strategy we discussed in the preceding section, it is important to optimise this measurement. This means obtaining the maximum unambiguous discrimination probability, or equivalently, the minimum probability of inconclusive results. To obtain this, we must know the a priori probabilities η_{\pm} of the two states. Given these, the total probability $P_{?}$ of obtaining an inconclusive result is

$$P_? = 1 - \sum_{j=+,-} \eta_j P_j. \tag{3.4}$$

The variational problem whose solution is $P_7(\text{opt})$ essentially consists of determining the values of P_{\pm} which minimise Eq. (3.4) subject to the constraint that the operator Π_7 is positive. For two states with equal a priori probabilities $\eta_{\pm} = 1/2$, it was established through the work of Ivanovic [22], Dieks [23] and Peres [24] that the minimum attainable value of the inconclusive result probability is given by

$$P_{7}(\text{opt}) = |\langle \psi_{+} | \psi_{-} \rangle|. \tag{3.5}$$

This Ivanovic-Dieks-Peres (IDP) limit is obtained when P_+ and P_- are both equal to $1 - |\langle \psi_+ | \psi_- \rangle|$, which is also the total probability of obtaining a conclusive, correct result. A more general bound was later obtained by Jaeger and Shimony [25], which solves the problem for unequal a priori probabilities. A particularly illuminating discussion of the Jaeger-Shimony result in the context of quantum communications was given by Ban [26]

It is important to understand what happens to the state of the system when an inconclusive result is obtained. It might be tempting to imagine that this outcome is of little importance, since we could repeat the measurement. Unfortunately, this is not the case. If an inconclusive result is obtained, then the states $|\psi_{\pm}\rangle$ undergo a transformation. In general, for a fixed pair of states, the lower $P_?$ is for the measurement, then the closer to each other, in terms of their overlap, the possible states will be after the transformation. As $P_?$ reaches

 $P_{?}(\text{opt})$ in Eq. (3.5), both states are transformed into the same state, rendering any further attempt to discriminate between them futile. Unambiguous discrimination can then be regarded as a kind of gambling operation. The states $|\psi_{\pm}\rangle$ are distinguishable to some extent, though not completely so. We can gamble this partial distinguishability in the hope of obtaining complete distinguishability, and will succeed with probability $1-P_{?}$. If we lose, however, the states become less distinguishable than they were initially.

As with the measurements described in the preceding section, photon polarisation states are ideally suited to the experimental realisation of this measurement [27]. The two states can be represented as

$$|\psi_{\pm}\rangle = \cos\theta |\uparrow\rangle \pm \sin\theta |\leftrightarrow\rangle.$$
 (3.6)

where again, $0 \le \theta \le \pi/4$. Consider now the interferometric setup depicted in Figure 6. A photon prepared in one of these states enters polarising beamsplitter PBS1. This is oriented so as to transmit photons which are horizontally polarised, and reflect the vertically polarised ones. The vertical polarisation component travels up to the ordinary beamsplitter BS, which has a transmission coefficient

$$t = \frac{\sqrt{\cos 2\theta}}{\cos \theta}.$$
 (3.7)

If a photon is transmitted here, then it will result in a count at detector $D_?$, and give an inconclusive result. The probability of this occurrence is the product of the probability that the photon is vertically polarised (to enable it to travel along the upper branch of the interferometer), which, from Eq. (3.6), is given by $\cos^2\theta$, and the probability of transmission, given by t^2 . The result is simply $\cos 2\theta = |\langle \psi_+ | \psi_- \rangle|$, which is the minimum probability in Eq. (3.5).

An inconclusive result is not obtained if this component is reflected at BS, in which case it will encounter a second polarising beamsplitter PBS2, which, like PBS1, transmits horizontal and reflects vertical polarisation. In the absence of a detection at PBS1, the photon will emerge from PBS2, its state having undergone the transformation

$$|\psi_{\pm}\rangle \rightarrow \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\leftrightarrow\rangle).$$
 (3.8)

These states are orthogonal, and can be distinguished using a third polarising beamsplitter PBS3. As in the Barnett-Riis experiment discussed in the preceding section, a polarising beamsplitter, here PBS3, oriented at $\pi/4$ to the horizontal can be used to deflect the photon to one of the detectors D_+ and D_- only when its initial state was $|\psi_+\rangle$ or $|\psi_-\rangle$ respectively. The wrong path is never taken, so that when successful, the discrimination attempt will always give the correct answer.

While technically feasible, the experimental apparatus shown in Figure 6 would have to be stabilised and aligned with extreme accuracy. This difficulty was overcome in an ingenious variation of this experiment reported in 1996 by Huttner et al [27]. An interferometer, such as that in Figure 6, or other device where photons with different polarisations travel along different paths, might seem to be essential for this type of experiment, since the nonorthogonal initial states are transformed into orthogonal ones using polarisation-dependent losses (PDL). What Huttner and collaborators realised was that the same effect can be achieved with all photons travelling along the same path if the medium through which they propagate itself has PDL. Using an optical fiber with this property, they performed the experiment using highly attenuated optical pulses (≈ 0.1 photons per pulse). For $\theta = \pi/6$, they obtained an error rate of 1.7%. Comparing this with the minimum error probability in the Helstrom measurement, which is approximately 6.7%, this measurement shows a clear improvement.

One disadvantage of using a PDL fiber is the fact that photons lost do not go to a detector, they simply do not register anywhere. Thus, the occasions when photons fail to result in a click at either D_+ or D_- , which we would like to interpret as being caused by inconclusive results, cannot be distinguished from those null events due to the weakness of the pulse. However, this does not detract from the fact that the error probability they obtained for detected photons was significantly less than the Helstrom bound.

Nevertheless, it is important to distinguish both kinds of null result. At the time of writing, an experimental realisation of the IDP measurement which follows the scheme shown in Figure 6 more closely, and produces the optimal theoretical proportions of conclusive and inconclusive results, has just been carried out by Clarke *et al* [28].

B. Unambiguous discrimination between linearly independent states

A naturally intriguing question is: how can this type of measurement be generalised to more than two states? We would then consider a set of N quantum states $|\psi_j\rangle$, with $j=1,\ldots,N$. Correspondingly, we would have detection operators Π_j satisfying

$$\langle \psi_{i'} | \Pi_i | \psi_{i'} \rangle = P_i \delta_{ii'}. \tag{3.9}$$

This condition says that outcome j can only occur when the initial state is $|\psi_j\rangle$. The conditional probability, given that the system was prepared in this state, that it will be successfully identified, is P_j . There will also be a further detection operator $\Pi_{?}$ corresponding to inconclusive results.

In [29], I showed that such a measurement strategy can only exist if the states $|\psi_j\rangle$ are linearly independent. For N linearly independent states, the form of the detection

operator Π_j is an immediate generalisation of that for the two state case, given by Eq. (3.2):

$$\Pi_j = \frac{P_j}{|\langle \psi_j^{\perp} | \psi_j \rangle|^2} |\psi_j^{\perp} \rangle \langle \psi_j^{\perp} |, \qquad \Pi_? = 1 - \sum_{j=1}^N \Pi_j. \quad (3.10)$$

The normalised state $|\psi_j^{\perp}\rangle$ is defined as that which is orthogonal to all $|\psi_{j'}\rangle$ for $j\neq j'$. Up to an overall phase, this state is unique and known as the *reciprocal state*. The relationship between the reciprocal states to the original states $|\psi_j\rangle$ is exactly analogous to that in crystallography between the reciprocal vectors and the Bravais lattice vectors, where each member of the former set of vectors is orthogonal to all but one member of the latter set [30].

As we saw in the case of unambiguous discrimination between just two states, it is important to examine how the possible initial states are transformed when an inconclusive result is obtained. Optimal unambiguous discrimination measurements on a pair of states transforms them into the same state if the measurement fails. It is shown in [29] that for a set of N linearly independent states, a failure will transform them into a linearly dependent set, making any further attempt to discriminate between them without errors impossible.

Nevertheless, an inconclusive result does not necessarily erase all of the information about the state (except in the case N=2), and it is still possible to obtain some information. If the unambiguous discrimination attempt fails, one can still carry out the quantum hypothesis testing strategy, described in the preceding section, on the resulting linearly dependent states.

An important question is whether or not one can obtain an analytic expression for the minimum probability of inconclusive results for more than two states. By analogy with the two state case, we take the state $|\psi_j\rangle$ to have a priori probability η_j , and see that the total probability of inconclusive results is given by

$$P_{?} = 1 - \sum_{j=1}^{N} \eta_{j} P_{j}. \tag{3.11}$$

As in the two state case, the variational problem consists of determining the P_j which minimise P_i subject to the constraint that the inconclusive result operator Π_i is positive.

Like the quantum hypothesis testing strategy, it is difficult to find explicit solutions for this optimisation problem for unambiguous discrimination with arbitrary states. However, also as with the hypothesis testing strategy described in the preceding section, the problem is explicitly soluble for equally probable symmetrical states, which satisfy Eqs. (2.23-2.24) [31]. In fact, at the time of writing, this is the only known solution for more than two states.

This solution is expressed in terms of a special representation of these states. Here, of course, we are concerned with symmetrical states which are also linearly in-

dependent. Note that these are completely distinct conditions. The trine ensemble of three symmetric states of a two-level system, discussed in the preceding section, is clearly linearly dependent. If the $|\psi_j\rangle$ are both linearly independent and symmetric, then they may be written as

$$|\psi_j\rangle = \sum_{k=1}^{N} c_k \exp\left(\frac{2\pi i j k}{N}\right) |k\rangle.$$
 (3.12)

for some coefficients c_k and orthonormal states $|k\rangle$. In fact, the $|k\rangle$ are the eigenstates of the unitary operator U in Eqs. (2.23) and (2.24). The coefficients c_k satisfy the normalisation condition $\sum_{k=1}^{N} |c_k|^2 = 1$. If the $|\psi_j\rangle$ have equal a priori probabilities $\eta_j = 1/N$, then the minimum value of P_i is given by

$$P_{?}(\text{opt}) = N \times \min_{k} |c_k|^2. \tag{3.13}$$

For optimum unambiguous discrimination between linearly independent symmetric states, it turns out that the conditional probabilities P_j are equal for all states, and therefore, from the equality of the a priori probabilities that we have been assuming, equal to $1-P_?(\text{opt})$. This measurement has been found to have some novel applications. For example, it has been shown by Dusek et al [32] that the possibility of such a measurement has worrying implications for quantum cryptography. These authors showed that for realistic implementations of the first quantum key distribution protocol, devised by Bennett and Brassard in 1984 (BB84), the use of this measurement as an eavesdropping strategy can render the protocol insecure for certain detector efficiencies.

The optimisation problem for more than two states has also been examined by Peres and Terno [33]. These authors gave a particularly detailed examination of the geometry and topology of the set of detection operators for 3 states, and showed how their method can be generalised to an arbitrary number of states.

In the following section, we shall look at a further interesting application of unambiguous discrimination between symmetric states, which relates to the manipulation of quantum entanglement.

IV. STATE DISCRIMINATION AND ENTANGLEMENT

A. Entanglement and quantum correlations

Recently, much attention has been paid to a peculiar type of correlation between quantum systems known as *entanglement*. In this section, we shall examine some of the main properties of entanglement, placing particular emphasis on those which are related to state discrimination. We shall begin by describing the type of nonlocal correlations which can occur between systems which are

entangled and show how, if it were possible to discriminate between arbitrary quantum states, then this could be used to transmit information across large distances instantaneously, in violation of the special theory of relativity.

Entanglement is produced when quantum systems interact with one another. If an operation on a pair of quantum systems involves no interaction between them, then it may be implemented as a series of distinct operations on the individual component systems, otherwise known as *local quantum operations*, perhaps together with classical communication between the agencies in possession of the components.

It is widely acknowledged that the fundamental properties of entanglement are that it is invariant under local unitary quantum operations and cannot, on average, increase under arbitrary local quantum operations and classical communication [34,35]. However, if we have a state which is slightly entangled, it is possible to act only on one of the subsystems in a way which sometimes produces more entanglement. Although using only local quantum operations, and possibly classical communication, we cannot increase entanglement on average, Bennett et al [36] discovered that we can gamble a small amount of initial entanglement with the possibility of obtaining more. This idea of obtaining, with some probability, a gain which cannot be acquired deterministically also lies at the heart of unambiguous discrimination. We shall see that the relationship between these two operations is far from superficial [29,37]. In fact, the local operation on one of the entangled subsystems which transforms the entire state into a maximally entangled state with maximum probability is also the operation which performs optimal unambiguous discrimination between a related set of symmetrical states [29], which we discussed in the preceding section.

It is helpful to begin by explaining what an entangled state is. Here, we shall consider only pure states. Suppose that Alice and Bob possess two quantum systems, A and B. If these systems have been prepared independently in the states $|\psi^1\rangle$ and $|\psi^2\rangle$ respectively, then the state of the combined system will be of the form

$$|\psi\rangle = |\psi^1\rangle_A |\psi^2\rangle_B. \tag{4.1}$$

Such a state is known as a *product state*. The significance of this form becomes apparent when we calculate expectation values of physical observables. Let α be an operator observable for A, and β be one for B. Then the expectation value of the product $\alpha\beta$ is simply

$$\langle \psi | \alpha \beta | \psi \rangle = \langle \psi^1 | \alpha | \psi^1 \rangle \langle \psi^2 | \beta | \psi^2 \rangle, \tag{4.2}$$

that is, it is simply the product of the expectation values of the two observables. If these two operators are projection operators, then they represent propositions, and their expectation values are the probabilities that these propositions are true. The product $\alpha\beta$, which is a projection operator on the space of the combined system,

represents the logical 'and' of these two propositions. We see then from Eq. (4.2) that the probability of ' α and β ' being true is simply the product of the probabilities of α being true and β being true. This implies that the truth probabilities of these two propositions are uncorrelated. Since these propositions are completely arbitrary, no property of particle A has any correlation with any of particle B.

The product state in Eq. (4.1) is not, however, the most general type of pure state of A and B. The superposition principle implies that the state of the entire system can be any linear combination of product states such as those in Eq. (4.1). An entangled state is such a superposition which cannot be expressed as a single product state. The product rule in Eq. (4.2) for expectation values of local observables does not generally hold for such states. One of the most extensively studied entangled states is the *singlet state*

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|u\rangle_A |u'\rangle_B - |u'\rangle_A |u\rangle_B).$$
 (4.3)

The states $|u\rangle$ and $|u'\rangle$ are orthogonal for each particle, and the singlet state is said to be maximally entangled. We shall shortly examine the problem of quantifying the amount of entanglement in general states. For the moment though, we shall consider only the singlet state as this can be used to show that, if it were possible to discriminate between arbitrary states, then information could be transmitted instantaneously.

We begin by noting that, since $|u\rangle$ and $|u'\rangle$ are orthogonal for each system, we can construct the Hermitian operators U_A and U_B which have these states as their eigenstates. A measurement of one of these operators can be used to distinguish perfectly between these states.

The singlet state has an interesting symmetry property, in that the states $|u\rangle$ and $|u'\rangle$ can be any pair of orthogonal states at all, and $|\psi\rangle$ still has this form shown in Eq. (4.3). We can then define another pair of orthogonal states, $|v\rangle$ and $|v'\rangle$ for each system, which are the eigenstates of the Hermitian operators V_A and V_B , and rewrite $|\psi\rangle$ as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|v\rangle_A |v'\rangle_B - |v'\rangle_A |v\rangle_B). \tag{4.4}$$

Measurements on entangled states such as $|\psi\rangle$ enforce non-local correlations between the subsystems. For example, if Alice measures U, then the state of her particle will collapse into either $|u\rangle$ or $|u'\rangle$. If Bob were then to carry out a measurement of U on his system, he would obtain, with unit probability, the opposite result, so that if Alice obtained $|u\rangle$, Bob would obtain $|u'\rangle$, and vice versa. The same holds true for the states $|v\rangle$ or $|v'\rangle$, or any other pair of orthogonal states. Alice is then able to predict the result of Bob's measurement, immediately, if he measures the same observable as her. If no signal can travel from Alice to Bob faster than the speed of light,

then we might be led to conclude that the information Alice obtains about Bob's subsequent measurement already exists in Bob's particle. However, since the observable is arbitrary, it would then follow that the information describing the results of all possible measurements on Bob's particle must already exist. Such an interpretation would no involve superluminal communication. This hypothesis, known as *local-realism*, is in sharp contrast to the idea of complementarity. Einstein, who discovered these correlations with Podolsky and Rosen [38] in 1935, expressed a preference for local realism over the alternative, which he imagined must be related to some kind of 'spooky action at a distance'.

In local-realistic theories, the apparent randomness of the results of quantum measurements is considered to be an illusion, perhaps due to our ignorance of some other significant parameters or hidden variables. It is then important to determine whether or not the predictions of quantum mechanics can be reproduced by a local-realistic hidden variable theory. However, in 1964, Bell published a theorem, according to which the correlations produced by any such theory must satisfy a certain inequality [39]. For suitable parameter choices, the predictions of quantum mechanics violate Bell's inequality for all pure entangled states, but not for product states [40], and curiously, not for some mixed entangled states [47]. The predictions of quantum mechanics have generally been vindicated by experiment, most famously in the experiments conducted by Aspect et al [42].

If these correlations are genuinely non-local, and do not result from the relativistically causal transmission of information between the two systems, can Alice and Bob use them to transmit information to each other? If Bob could discriminate, with zero probability of error, between the four states $|u\rangle$, $|u'\rangle$, $|v\rangle$ or $|v'\rangle$, then he could tell whether Alice measured U_A or V_A . If she wishes to transmit '1' to Bob, she measures U. If on the other hand, she wishes to communicate a '0' to him, she would measure V_A instead.

The nonlocal nature of entanglement cannot be used to transmit information in this manner, which would avoid the actually sending of physical systems, and thus the universal speed limit c. General proofs of the impossibility of superluminal communication using entanglement and measurement have been obtained [43]. On the basis of the above argument, these proofs must implicitly place restrictions on the extent to which the state of a quantum system can be determined.

Although entanglement cannot be used for superluminal communication, it does have several other applications. For example, it can be used, in conjunction with a classical communication channel, to teleport an unknown quantum state from one location to another [44]. It can also be used to transmit classical information at twice the maximum rate that can be achieved using classical physics, using a technique known as superdense quantum coding [45]. Also, many of the recently discovered advantages of using quantum systems for computing (the best

known of which is Shor's algorithm for factorising a number in polynomial time [7], for which the best known classical algorithms require exponential time) make explicit use of entanglement to carry out computations more efficiently than any computer operating solely by the laws of classical physics can manage.

It is therefore important to understand the conditions under which entanglement can be manipulated. In particular, how do we quantify entanglement? If Alice and Bob share some entangled state $|\psi\rangle$, how much entanglement does it contain?

To answer this, we have to understand the most general form of an entangled state. We arrived at the concept of entanglement via the superposition principle, which enabled us to construct a linear combination of product states which is not itself a product state. The most general pure state of a two-particle system is simply the most general superposition of product states. For a pair of N-level quantum systems, this is

$$|\psi\rangle = \sum_{j,k=1}^{N} b_{jk} |\alpha_j\rangle_A |\beta_k\rangle_B, \tag{4.5}$$

where, without loss of generality, the subsystem states $|\alpha_j\rangle$ and $|\beta_k\rangle$ are taken to be orthogonal, so that $\sum_{i,k=1}^{N} |b_{ik}|^2 = 1$.

 $\sum_{j,k=1}^{N} |b_{jk}|^2 = 1$. The b_{jk} are almost completely free parameters. They are subject only to this normalisation constraint. Such a large number of free parameters makes the expression in Eq. (4.5) somewhat unwieldy. Fortunately, a simpler representation of $|\psi\rangle$ can be obtained using an important result known as the *Schmidt decomposition theorem* [46]. This states that there exists an orthogonal basis $|\alpha'_j\rangle$ for particle A and $|\beta'_j\rangle$ for particle B such that the state $|\psi\rangle$ takes the form

$$|\psi\rangle = \sum_{j=1}^{N} c_j |\alpha'_j\rangle_A |\beta'_j\rangle_B, \qquad \sum_{j=1}^{N} |c_j|^2 = 1, \qquad (4.6)$$

i.e. using these special bases, known as the *Schmidt bases*, we can write the state as a single, rather than a double sum over product states.

For a product state, only one of the c_j is non-zero. If more than one is non-zero, the state is entangled. To quantify this entanglement, we have to form the density operator for one of the subsystems. These are known as reduced density operators, and are denoted by ρ_A and ρ_B . The reduced density operator of either subsystem is formed by taking the trace of density operator of the entire system with respect to the other subsystem, i.e.

$$\rho_A = \operatorname{Tr}_B(|\psi\rangle\langle\psi|), \quad \rho_B = \operatorname{Tr}_A(|\psi\rangle\langle\psi|).$$
(4.7)

We find that these are

$$\rho_A = \sum_{j=1}^{N} |c_j|^2 |\alpha_j'\rangle \langle \alpha_j'|, \qquad \rho_B = \sum_{j=1}^{N} |c_j|^2 |\beta_j'\rangle \langle \beta_j'|. \quad (4.8)$$

We are now in a position to quantify the amount of entanglement in the state $|\psi\rangle$. The entanglement, or entropy of entanglement, $E(\psi)$, is given by the *von Neumann entropy* [47] of either of the reduced density operators:

$$E(\psi) = -\text{Tr}_A \rho_A \log \rho_A = \text{Tr}_B \rho_A \log \rho_B$$
$$= -\sum_{j=1}^N |c_j|^2 \log |c_j|^2. \tag{4.9}$$

The logarithm is conventionally taken to have base 2. Entanglement is measured in *ebits*. If only one c_j is nonzero, corresponding to a product state, then $E(\psi)=0$. At the other extreme, if all $|c_j|^2$ are equal, then normalisation implies that $|c_j|^2=1/N$, and $E(\psi)=\log N$. Such states are called *maximally entangled states* since they possess the most entanglement for a given N. The singlet state in Eq. (4.3) is a maximally entangled state, and has 1 ebit of entanglement.

One of the major advantages of using the entropy of entanglement to quantify this property is the fact that it is additive. To understand the meaning of this, suppose that Alice and Bob share 2 entangled states, $|\psi\rangle$ and $|\psi'\rangle$. How much entanglement do they possess? Considering these two entangled systems individually, we would conclude that the total entanglement shared by Alice and Bob is just the sum of the entanglements of $|\psi\rangle$ and $|\psi'\rangle$. If, however, we consider them to be composite parts of a larger entangled system in the state $|\psi_L\rangle = |\psi\rangle|\psi'\rangle$, then the entanglement shared by Alice and Bob is that in $|\psi_L\rangle$. Clearly, to quantify shared entanglement unambiguously, it is necessary that an entanglement measure E satisfies $E(\psi_L) = E(\psi) + E(\psi')$. Fortunately, the entropy of entanglement has this desirable additive property for pure entangled states. However, no entanglement measure has vet been shown to be additive over the set of all mixed entangled states. Recently, though, additivity has been shown to hold for one of the most important entanglement measures, the relative entropy of entanglement, for a large class of states [48].

B. Entanglement concentration and unambiguous state discrimination

The entanglement $E(\psi)$ cannot be deterministically increased by acting on the subsystems individually, even if classical communication is allowed between Alice and Bob. However, it is possible to use an unambiguous discrimination-type measurement on either A or B which will, with some probability, convert $|\psi\rangle$ into a maximally-entangled state. Such an operation is known as entanglement concentration. For the sake of definiteness, we let the measurement be performed by Alice on particle A. We first make use of a new orthogonal basis set $|y_k\rangle$ for Bob's particle. These states are defined through

$$|\beta_j'\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left(\frac{2\pi i j k}{N}\right) |y_k\rangle.$$
 (4.10)

This expression allows us to rewrite the partly entangled state $|\psi\rangle$ in Eq. (4.6) as

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |x_k\rangle_A |y_k\rangle_B, \tag{4.11}$$

where we have introduced another set of new states $|x_k\rangle$, defined by

$$|x_k\rangle = \sum_{j=1}^{N} c_j \exp\left(\frac{2\pi i j k}{N}\right) |\alpha_j'\rangle.$$
 (4.12)

Both the orthonormality of the $|y_k\rangle$ and the representation in Eq. (4.11) follow from the relation

$$\sum_{k=1}^{N} \exp\left(\frac{2\pi i (j-j')k}{N}\right) = N\delta_{jj'}.$$
(4.13)

The $|x_k\rangle$ are normalised, although they are not orthogonal. If they were, then the expression in Eq. (4.11) for $|\psi\rangle$ would represent a maximally-entangled state, which it cannot be since our transformation of basis is passive, doing nothing to change the entanglement of the state. These states are, however, linearly independent. In fact, comparing Eq. (4.12) with Eq. (3.12), we see that they constitute a set of linearly independent symmetric states of the kind we discussed in the preceding section.

Looking at Eq. (4.11), we can say that what prevents $|\psi\rangle$ from being maximally entangled is the nonorthogonality of the $|x_k\rangle$. However, unambiguous discrimination can be regarded as an operation which transforms non-orthogonal states into orthogonal ones. This was made explicit in the experimental realisation of unambiguous discrimination for a pair of states [27,28], discussed in the preceding section. To see how this can be exploited to transform $|\psi\rangle$, with some probability, into a maximally-entangled state, consider the detection operators for unambiguous discrimination between the $|x_k\rangle$. It follows from Eq. (3.10) that these are $\Pi_k = P_k |x_k^{\perp}\rangle \langle x_k^{\perp}|/|\langle x_k^{\perp}|x_k\rangle|^2$. The states $|x_k^{\perp}\rangle$ are the reciprocal states corresponding to the $|x_k\rangle$. These operators, together with $\Pi_? = 1 - \sum_{k=1}^N \Pi_k$, represent a measurement whose possible outcomes are the N states $|x_k\rangle$, and the inconclusive result. To use this type of measurement for entanglement concentration, it is more appropriate to consider just two outcomes, described by the inconclusive result operator, $\Pi_{?}$, and the following operator

$$\Pi_O = \sum_{k=1}^{N} \Pi_k. \tag{4.14}$$

By definition, this pair of operators forms the required resolution of identity in Eq. (2.13). The reason for subscript O in Eq. (4.14) will become apparent shortly. This

pair of detection operators describes a measurement having two outcomes: success or failure of the state discrimination measurement. It does not tell us which state has been detected when it succeeds. This might not seem very useful at first sight. However, the state transformation generated by this measurement is precisely that which transforms $|\psi\rangle$ into a maximally entangled state. As we discussed in section II, to determine how the state of a system is transformed by a generalised measurement, we need to find an operator A such that $\Pi = A^{\dagger}A$, when the result corresponds to the detection operator A. For the operator A0, the corresponding transformation operator is the orthogonalisation operator A0

$$A_O = \sum_{k=1}^{N} \frac{P_k^{1/2}}{\langle x_k^{\perp} | x_k \rangle} |\phi_k\rangle \langle x_k^{\perp}|, \tag{4.15}$$

where the states $|\phi_k\rangle$ may be any orthogonal basis. The arbitrariness of this basis is equivalent to that of the unitary transformation in Eq. (2.14). As we saw in the preceding section, the maximum probability of discriminating between a set of N symmetric states is obtained when all of the P_k are equal to $1-P_?(\text{opt})$, where $P_?(\text{opt})$ is given by Eq. (3.13).

Using the prescription for state transformations in Eq. (2.15), we see that if the system is initially prepared in one of the non-orthogonal states $|x_k\rangle$, when this measurement succeeds, then the state will be transformed into the corresponding member of the orthogonal basis $|\phi_k\rangle$. In fact, the unambiguous discrimination measurement can be regarded as this orthogonalisation procedure, followed by a von Neumann measurement in the orthogonal basis $|\phi_k\rangle$. This is precisely what happens in the photonic implementation of two-state discrimination we discussed in section III. There, the non-orthogonal photon states were, with probability $1 - P_?(\text{opt})$, transformed into the orthogonal states in Eq. (3.8) before being discriminated.

Looking at the representation we have for the entangled state $|\psi\rangle$ in Eq. (4.11), we see that if Alice carries out this operation on particle A, and if it succeeds, then the final state will be

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} |\phi_k\rangle_A |y_k\rangle_B, \tag{4.16}$$

which is our promised maximally-entangled state. The probability of success for this operation is given by $1 - N \times \min |c_j|^2$. As it happens, this is the maximum probability of converting $|\psi\rangle$ into a maximally-entangled state. The maximum probability of converting a non-maximally entangled pure state of two systems into a maximally-entangled one, using only local quantum operations and possibly classical communication, was found by Lo and Popescu [49]. The strategy we have been discussing reaches their bound.

Several additional results on transforming one entangled state into another using only local operations and

classical communication have been found. An important question is: under what circumstances can one pure entangled state be transformed into another with unit probability? The solution to this problem was obtained by Nielsen [50], whose results introduced the powerful mathematical technique of majorisation to the study of entanglement.

Deterministic transformations of this kind are not possible for all pairs of states. Vidal [51] obtained a general expression for the maximum probability that any pure entangled state can be converted into any other.

V. UNAMBIGUOUS DISCRIMINATION AND EXACT CLONING

A. Relationship between quantum cloning and state discrimination

Another operation closely related to unambiguous discrimination is exact cloning. In 1982, it was discovered independently by Wootters and Zurek [52] and Dieks [53] that the state of a quantum system, if unknown, cannot be copied. As with discrimination, no completely reliable procedure exists for this unless the state belongs to a known orthogonal set.

By analogy with the possibility of unambiguous discrimination between linearly independent states, it is possible, as was initially demonstrated by Duan and Guo [56], to build a machine which, with some probability, produces exact copies of such states. We will examine the relationship between these two operations, in particular, that between their maximum success probabilities. In fact, for just two states, both operations can be regarded as particular cases of a more general procedure known as quantum state separation [57], which we will also describe.

To understand the relationship between state discrimination and cloning, suppose that Alice gives Bob one of the N quantum states $|\psi_j\rangle$. He isn't told which, although again, he knows what the possible states $|\psi_j\rangle$ are, and also their a priori probabilities η_j . If he can discriminate between them, then upon identifying the state, he can manufacture as many further copies of it as he desires. Therefore, if he can discriminate between them, then he can also clone them.

If, on the other hand, Bob could clone a set of states, then he could also discriminate between them. This follows from the fact that if Bob could make one copy, he could make arbitrarily many. He could then make use of the fact that, given a sufficiently large number of copies of the state, he could determine the expectation value of any observable, to an arbitrarily high degree of accuracy, by repeatedly measuring it on the members of his ensemble of clones. If the ensemble is large enough, he could evaluate the expectation values of several observables, in

fact, any finite number of them, to any degree of accuracy. If he chooses the correct observables, it would be possible for him to infer the state itself from the expectation values. For an N-dimensional system, the density operator is specified by N^2-1 independent real parameters. It is quite easy to see why. The density operator has N^2 elements, each of which is complex. It is therefore determined by $2N^2$ real parameters. There are N^2 constraints due to Hermiticity, and a further constraint comes from the requirement of normalisation. Therefore, to determine the state, Bob must know the expectation values of at least N^2-1 observables. Fortunately, with a judicious choice of observables, this lower bound can be attained.

The simplest example is the case of a two-level system or qubit. The state ρ of a qubit can be conveniently expressed in the Bloch representation. We are already familiar with the representation of a qubit as a spin-1/2 particle, and the eigenstates $|\pm\rangle$ of the z component of the spin, this being represented by the operator σ_z . It remains to introduce the other Cartesian components σ_x and σ_y of this vector operator, the Pauli spin operator σ . These component act on the eigenstates of σ_z in the following way:

$$\sigma_x |\pm\rangle = |\mp\rangle, \qquad \sigma_y |\pm\rangle = \pm i |\mp\rangle.$$
 (5.1)

All three Cartesian components of σ are both Hermitian and unitary operators (implying that the square of each of them is 1). They also have eigenvalues ± 1 , so that their trace is zero. The Bloch representation of the density operator ρ is obtained by writing it as a combination of these operators and the identity,

$$\rho = \frac{1}{2}(1 + \mathbf{a}.\boldsymbol{\sigma}). \tag{5.2}$$

The components of the Bloch vector \mathbf{a} are real and the length of this vector $|\mathbf{a}|$ is no greater than 1. If it is equal to 1, then ρ is a pure state. If it is equal to 0, then $\rho = 1/2$, meaning that it is a completely mixed state. Thus, we may take the length of the Bloch vector to be an indicator of how pure the state is.

Determination of the state ρ of a qubit clearly amounts to finding the components of the Bloch vector, whose 3 components are a special case of the general number of parameters N^2-1 . To evaluate these components, Bob need only measure the expectation values of each σ_k . These are equal to $\text{Tr}\sigma_k\rho=a_k$, where k=x,y,z. This can be seen from the fact that σ_k has zero trace, and from the identity $\text{Tr}\sigma_k\sigma_l=2\delta_{kl}$.

While it is useful to know that, in principle, the expectation values of a set of observables are sufficient to determine the state of a system, in practice, these quantities cannot be measured exactly, as this would require an infinite number of measurements to be carried out. Derka et al [54] showed how the state can nevertheless be estimated from the available data using Bayesian inference techniques.

Using standard von Neumann measurements, the state can be extracted from a large number of copies by measuring $N^2 - 1$ observables. An interesting question is: can this number of types of measurement be reduced if we use generalised measurements instead? This issue was explored by Peres and Terno [55], who came to the intriguing conclusion that only a single generalised measurement is necessary. Their strategy is essentially as follows: consider a generalised measurement with N^2-1 detection operators Π_k . If we have a large number of systems, all prepared in the same unknown state ρ , then we can determine the probability of the kth outcome, ω_k , which, by Eq. (2.11), is given by $P(\omega_k|\rho) = \text{Tr}\rho\Pi_k$. What Peres and Terno realised was that there exist generalised measurements for which the N^2-1 probabilities $P(\omega_k|\rho)$ are one possible set of parameters which can be used to infer the density operator itself.

From the above discussion, it is apparent that if Bob can copy the state he has, then he can determine it, and vice versa. Unambiguous state discrimination is possible only if the state belongs to a known, linearly independent set. On the basis of the above argument, we should expect that the same constraint limits the abilities of cloning machines. This is indeed the case. It was recently discovered by Duan and Guo [56] that only linearly independent states can be cloned, and only with unit probability if they are orthogonal.

B. Exact cloning and unambiguous state discrimination

An intriguing question is the following: given this symbiotic relationship between discrimination and cloning, the essence of which is that the conditions under which one of these operations is possible also apply to the other, are there quantitative relationships between their optimal figures of merit? On the basis of the relationship between unambiguous discrimination and exact cloning, we would expect their maximum success probabilities to be related.

We will see how to obtain a bound on the maximum probability of cloning two equally probable states using the bound on the probability of discriminating between two states, given by the Ivanovic-Dieks-Peres limit in Eq. (3.5). We then show that this cloning bound also leads to the IDP bound. Finally, we will show that both bounds are special cases of a more general limit which relates to an operation known as quantum state separation.

For two equally-probable pure states $|\psi_{\pm}\rangle$, the maximum probability of success for unambiguous discrimination is given $1 - |\langle \psi_{+} | \psi_{-} \rangle|$, from the IDP limit in Eq. (3.5). It follows from this that if Bob has M copies of the system, then the maximum probability of discriminating between these M copies $|\psi_{\pm}\rangle_{1} \dots |\psi_{\pm}\rangle_{M}$ is

$$P_{M\infty}(\text{opt}) = 1 - |\langle \psi_+ | \psi_- \rangle|^M. \tag{5.3}$$

The use of the notation $P_{M\infty}$ will become apparent shortly. If this is the maximum probability with which one can discriminate between these M-particle states, then it is impossible to improve upon this bound by the following procedure. We first attempt an N from M cloning operation, that is, to transform these M copies, together with N-M particles in 'blank states', into N copies, where $N{\geq}M$. If this succeeds, we then attempt to discriminate between the N-particle products, which cannot be accomplished with probability greater than

$$P_{N\infty}(\text{opt}) = 1 - |\langle \psi_+ | \psi_- \rangle|^N. \tag{5.4}$$

The cloning probability, which we shall write as P_{MN} , must be constrained by the fact that this compound operation cannot be accomplished with probability greater than $P_{M\infty}$ (opt). If this were not true, then $P_{M\infty}$ could not be the maximum probability of distinguishing between M copies of $|\psi_{+}\rangle$ or $|\psi_{-}\rangle$. Thus, $P_{M\infty}$ (opt) $\geq P_{MN}P_{N\infty}$ (opt). In fact, it was shown in [57] that the equality here can be attained, implying that the maximum cloning probability is

$$P_{MN}(\text{opt}) = \frac{1 - |\langle \psi_{+} | \psi_{-} \rangle|^{M}}{1 - |\langle \psi_{+} | \psi_{-} \rangle|^{N}}.$$
 (5.5)

This generalises an earlier result by Duan and Guo [58] that the maximum probability of making two copies of the state given one initially is

$$P_{12}(\text{opt}) = \frac{1}{1 + |\langle \psi_+ | \psi_- \rangle|}.$$
 (5.6)

Exact cloning in this manner has not yet been realised in the laboratory, although a quantum-computational network which achieves this task has been proposed [59]. A further recent development in the study of probabilistic cloning machines is that it is possible for the actual number of copies to be a quantum, rather than a classical variable. Pati [60] has shown how one can construct a 'novel' cloning machine which, with some probability, will generate a superposition of various numbers of exact copies.

We see that the bound on the maximum probability of unambiguous discrimination implies a corresponding bound on the maximum probability of exact cloning. As it happens, the bound on exact cloning $P_{MN}(\text{opt})$ also implies that $P_{1\infty}(\text{opt})$ in Eq. (5.3) is actually the maximum probability of unambiguously discriminating between the states $|\psi_{\pm}\rangle$.

Given one initial copy of the state, the maximum probability that we can make N copies is given by $P_{1N}(\text{opt})$. We can see from Eq. (5.5) that as $N \to \infty$, $P_{1N}(\text{opt}) \to 1 - |\langle \psi_+ | \psi_- \rangle|$ from above. In this limit, the state could be inferred through the statistics of appropriate measurements on the copies, so we have shown how Eq. (5.5) implies that the states can be discriminated unambiguously with probability $P_{1\infty}(\text{opt})$. Consistency with the cloning bound implies that no greater value than

 $P_{1\infty}(\text{opt})$ can be attained. If state discrimination could be accomplished with probability higher than $P_{1\infty}(\text{opt})$, then we could, with the same probability, make an arbitrarily large number of copies of the state given one initial realisation. If this probability was greater than $P_{1\infty}(\text{opt})$, it would also exceed P_{1N} for sufficiently large N. Therefore, the discrimination bound can also be obtained from the cloning bound.

C. Quantum state separation

We will conclude this section with a brief discussion of a general quantum operation which has unambiguous discrimination and exact cloning as special cases. This operation is known as *quantum state separation*.

Consider two non-orthogonal quantum states $|\phi_{\pm}^1\rangle$. We would like to know the maximum probability with which these can be transformed into another pair of quantum states $|\phi_{\pm}^2\rangle$ such that

$$|\langle \phi_{+}^{2} | \phi_{-}^{2} \rangle| < |\langle \phi_{+}^{1} | \phi_{-}^{1} \rangle|. \tag{5.7}$$

The (modulus of the) overlap of the final states is less than that of the initial states, hence the term state separation. In [57], it is shown that if both states have equal a priori probabilities, then the maximum value $P_S(\text{opt})$ of this separation probability is

$$P_S(\text{opt}) = \frac{1 - |\langle \phi_+^1 | \phi_-^1 \rangle|}{1 - |\langle \phi_+^2 | \phi_-^2 \rangle|}.$$
 (5.8)

This operation corresponds to unambiguous discrimination when the final states $|\phi_{\pm}^2\rangle$ are orthogonal. In this case, the denominator is equal to 1, and $P_S(\text{opt})$ is equal to the Ivanovic-Dieks-Peres limit on the probability of conclusively distinguishing between the states $|\phi_{\pm}^1\rangle$.

Suppose instead that $|\phi_{\pm}^1\rangle$ represents M copies of the state $|\psi_{\pm}\rangle$, together with N-M particles in some collective 'blank' state $|\chi\rangle$. We then have $|\phi_{\pm}^1\rangle = |\psi_{\pm}^1\rangle_1...|\psi_{\pm}^1\rangle_M|\chi\rangle$. We take the final states $|\phi_{\pm}^2\rangle$ to be N copies of the state $|\psi_{\pm}\rangle$, that is, $|\phi_{\pm}^2\rangle = |\psi_{\pm}^2\rangle_1...|\psi_{\pm}^2\rangle_N$. The modulus of the overlap between the final states, $|\langle\phi_{+}^2|\phi_{-}^2\rangle| = |\langle\psi_{+}|\psi_{-}\rangle|^N$, is less than that of the corresponding initial states, $|\langle\phi_{+}^1|\phi_{-}^1\rangle| = |\langle\psi_{+}|\psi_{-}\rangle|^M$. Exact cloning is then a further example of state separation. Substitution of these expressions into Eq. (5.8) gives the maximum cloning probability $P_{MN}(\text{opt})$ in Eq. (5.5).

In our discussion of state discrimination, we saw the importance of examining what happens to the state of the system when the operation fails. The erasure of information that takes place there also occurs for the more general state separation operation, and therefore also for cloning. When an optimal state separating operation fails, the possible initial states $|\phi_{\pm}^1\rangle$ are transformed into the same state, rendering any further separation attempt impossible.

VI. UNIVERSAL STATE ESTIMATION AND CLONING

A. Estimating a completely unknown state

So far, we have been examining the problem of trying to discriminate, as best as we can, between members of a known, finite set of states. Suppose that the state is completely unknown. In the two level case, Alice might give Bob a qubit prepared in the state

$$|\psi\rangle = a|+\rangle + b|-\rangle,\tag{6.1}$$

and Bob has no information at all about the values of the coefficients a and b (except, of course, that $|a|^2 + |b|^2 = 1$, for normalisation). The strategies we have examined for finite sets are not useful here. The set of possible states is clearly linearly dependent, so unambiguous discrimination between them is impossible. More generally, any realistic detection strategy will have a finite number of outcomes. Since we are dealing here with an infinite set of possible states, we cannot uniquely associate one outcome with every possible state, even if we allow for errors. Massar and Popescu [61] and Derka, Bužek and Ekert [62] examined this problem from a different perspective, proposing a more realistic strategy known as quantum state estimation. The problem can be formulated as a game. Alice gives Bob M copies of the state $|\psi\rangle$, and his task is to perform a measurement with K outcomes. On the basis of the outcome he records, he will conjecture that the state was a member of some finite set of states $|\omega_k\rangle$, where $k=1,\ldots,K$. In general, his guess will be wrong, and the idea is to construct the measurement such that the conjectured state is, on average, as close as possible to the actual state.

The accuracy of Bob's guess is measured by a score function. Slightly different score functions are chosen by the two sets of authors, although their end results are the same. The experimental significance of their functions can be appreciated if the qubits are realised as photon polarisation states. Massar and Popescu take the score function to be $\cos^2(\alpha/2)$, where α is the angle between the actual and guessed directions of polarisation. Derka et~al~ use $\cos^2\alpha$. This latter choice has a significance for general quantum systems. It is known as the fidelity. If the actual state of the system is $|\psi\rangle$ and the guessed state is $|\omega\rangle$, then the fidelity $F(\omega|\psi)$ is simply the square-overlap between them,

$$F(\omega|\psi) = |\langle \omega|\psi\rangle|^2. \tag{6.2}$$

The fidelity has the following interpretation. Consider a measurement designed to determine whether or not a quantum system has been prepared in the state $|\omega\rangle$. The best measurement is a so-called maximal measurement. This is a von Neumann measurement of an operator observable which has $|\omega\rangle$ among its eigenstates. If the initial state of the system is $|\psi\rangle$, then the fidelity F in Eq. (6.2)

is the probability that the result of this measurement is 'yes'. The fidelity is then a natural and practical measure of how closely the states $|\omega\rangle$ and $|\psi\rangle$ resemble each other.

In the context of polarisation, the significance of the fidelity can be appreciated by considering Malus' Law in its photonic form. This tells us that the probability that the actual photon state would pass through a polariser designed to transmit photons in the guessed polarisation state is $\cos^2 \alpha$, i.e., it is equal to the fidelity.

The half-angle formula of Massar and Popescu is also useful for the following reason: if the fidelity score is used, then when the real and guessed polarisations are orthogonal, we would obtain a score of 0. However, this corresponds to as much an information gain as in cases when the score reaches 1. This is because a fidelity of 0 is only obtained when the guess state $|\omega\rangle$ and the actual state $|\psi\rangle$ are orthogonal. A fidelity of 1 could then be obtained by replacing each guess state by the state orthogonal to it. However, this property presents no problems if we are interested in maximising the score function, averaged over all states, which we shall be. Nevertheless, Massar and Popescu's half-angle score function avoids this ambiguity. Its minimum value is 1/2, which corresponds to the actual state having probability of 1/2 of being in either the guessed state or the one orthogonal to it, which means no information gain at all.

The best measurement for any score function is that which maximises the average of the score over all states. As it happens, the maxima of both score functions are equal, although for the sake of concreteness, we shall concentrate on the fidelity. The average fidelity, given M initial copies of the unknown state $|\psi\rangle$, is

$$\bar{F}_M = \sum_{k=1}^K \int \mathcal{D}|\psi\rangle P(\omega_k|\psi) F(\omega_k|\psi), \tag{6.3}$$

where $P(\omega_k|\psi)$ is the probability that the guessed state is $|\omega_k\rangle$ given that the actual state is $|\psi\rangle$. This function resembles the Bayes' cost function in Eq. (2.4), although the positive nature of a 'score' contrasts with the negative nature of a cost, which implies that it is desirable to maximise Eq. (6.3), whereas we would prefer to minimise the Bayes' cost C_B in Eq. (2.4). This distinction, however, is quite superficial: it amounts merely to a difference of sign. There are more significant differences between these two figures of merit. One of the most obvious differences between C_B and F_M is that the latter refers to a continuous set of states, while the former refers to a discrete set. Also, in evaluating the Bayes' cost, the possible states will, in general, have different a priori probabilities. In quantum state estimation however, we have no a priori information about the state of the system, so the a priori probability density is uniform. Perhaps the most pertinent difference between the two strategies is that, in hypothesis testing, the number of outcomes is fixed: it is equal to the number of possible states. In contrast, there is no a priori information about the guess states $|\omega_k\rangle$ in state estimation. All properties of the guess states, including how many of them there are, are to be determined through the optimisation procedure, that is, the maximisation of \bar{F}_M .

Given M initial copies of the state, Derka et~al showed that the maximum value of \bar{F}_M is

$$\bar{F}_M(\text{opt}) = \frac{M+1}{M+2}.$$
 (6.4)

Massar and Popescu showed that this is also the maximum of \bar{F}_M if $F(\omega_k|\psi)$ is replaced by $\cos^2\alpha/2$. We see that the maximum score increases with M until, in the limit as $M\to\infty$, it attains the value of 1.

Massar and Popescu proved the existence of a finite set of guess states which attains the optimum score. Derka et al provided an explicit algorithm for finding these states. Actually, their algorithm gives the optimum measurement for any set of states generated from some reference state $|\psi_0\rangle$ by a unitary, finite representation of a compact Lie group. The Massar-Popescu paper is concerned with the group SU(2), that is, the group which generates all possible pure states of a qubit from the reference state.

If Alice gives to Bob M copies of the state $|\psi\rangle$, then he will obtain the guess state $|\omega_k\rangle$ with probability $P(\omega_k|\psi)$. What state then does Bob guess 'on average'? It is simply a mixture of the guess states weighted by their respective probabilities, which we denote by ρ_{quess} :

$$\rho_{guess} = \sum_{k=1}^{K} P(\omega_k | \psi) |\omega_k\rangle \langle \omega_k|.$$
 (6.5)

An appropriate measure of how closely this average guess state resembles the actual state ρ_{guess} is again given by the fidelity:

$$F_M(\psi) = \langle \psi | \rho_{guess} | \psi \rangle = \sum_{k=1}^K P(\omega_k | \psi) F(\omega_k | \psi). \quad (6.6)$$

The optimum mean fidelity $\bar{F}_M(\text{opt})$ in Eq. (6.4) is just the average of the optimum $F_M(\psi)$ over all states $|\psi\rangle$. The symmetry of the optimal measurement implies that $F_M(\psi)$ must actually be independent of $|\psi\rangle$. This implies that the average guess state ρ_{quess} must have the form:

$$\rho_{guess} = \frac{1}{2}(1 - S_M) + S_M |\psi\rangle\langle\psi|. \tag{6.7}$$

where $0 \le S_M \le 1$. The quantity S_M is known as a *shrink-ing factor*, for the following reason. Let us write the density operator $|\psi\rangle\langle\psi|$ in the Bloch representation described in the preceding section: that is, $|\psi\rangle\langle\psi| = (1 + \mathbf{a}.\boldsymbol{\sigma})/2$, for some unit vector \mathbf{a} . Then one can show using Eq. (6.7) that ρ_{quess} has the Bloch representation

$$\rho_{guess} = \frac{1}{2} (1 + S_M \mathbf{a}.\boldsymbol{\sigma}). \tag{6.8}$$

The shrinking factor S_M decreases the length of the Bloch vector \mathbf{a} , although it's direction in maintained. If $S_M =$

1, then ρ_{guess} is equal to the original state $|\psi\rangle\langle\psi|$. On the other hand, if $S_M=0$, then ρ_{guess} is completely mixed, and contains no information about the initial state.

The optimal state estimation strategy maximises the average fidelity \bar{F}_M . It also maximises the shrinking factor S_M . This can be seen in the following way. The shrinking factor is independent of the actual state $|\psi\rangle$. Therefore, $F_M(\psi)$ in Eq. (6.6) is simply equal to $(1+S_M)/2$. Substituting this into Eq. (6.3) shows that \bar{F}_M must have the form constant× $(1+S_M)/2$. However, this constant is unity, due to normalisation of the integral. Thus, we have

$$\bar{F}_M(\text{opt}) = \frac{1}{2}(1 + S_M(\text{opt})),$$
 (6.9)

leading to

$$S_M(\text{opt}) = \frac{M}{M+2}. (6.10)$$

If only one copy of the state is initially available, then the maximum value of the shrinking factor is 1/3, and the Bloch vector is reduced to 1/3 of its former length. However, the shrinking factor grows with increasing M until, in the limit as $M \rightarrow \infty$, the shrinking factor tends to unity.

B. Universal cloning machines

As with unambiguous discrimination, the optimal figure of merit in Eq. (6.4) for universal state estimation is intimately related to the optimal efficiency of cloning. In this case, it is the optimal efficiency of universal cloning. The idea of a universal quantum cloning machine (UCM) was conceived by Hillery and Bužek [63].

The idea is essentially this: Alice gives Bob M copies of a quantum system prepared in some state $|\psi\rangle$. All states are equally probable. Now, we know that only linearly independent states can be cloned exactly, so the copies produced by such a cloning machine will necessarily be imperfect. The degree of imperfection of the clones is most easily expressed using the Bloch representation. Universal cloning machines are designed to copy all states equally well, and are thus of a highly symmetrical nature, If all of the clone states are required to be identical, one of the consequences is that the direction of the Bloch vector is identical to that of the original state. However, its length decreases by a shrinking factor S_{MN} which is independent of the state cloned and depends only upon M, the number of initial, exact copies and N, the number of final approximate copies. If the initial copies are of the form shown in Eq. (5.2), then the output ones look like

$$\rho_{out} = \frac{1}{2} (1 + S_{MN} \mathbf{a}.\boldsymbol{\sigma}). \tag{6.11}$$

Optimising a universal cloning machine means minimising the decrease in the length of the Bloch vector. Bruss

et al [64] showed that the optimum, that is, the maximum value of this shrinking factor is

$$S_{MN}(\text{opt}) = \frac{M(N+2)}{N(M+2)}.$$
 (6.12)

As in universal state estimation, the performance of a UCM can be characterised by either the shrinking factor or the fidelity between the actual state and the state obtained. We shall denote by F_{MN} the fidelity between the output state of a UCM and the exact state $|\psi\rangle$, given that M copies of $|\psi\rangle$ were supplied to the UCM, which then produced N>M imperfect copies. As a consequence of symmetry, F_{MN} , like the shrinking factor, is independent of $|\psi\rangle$. From the definition $F_{MN}=\langle\psi|\rho_{out}|\psi\rangle$, we easily find that $F_{MN}=(1+S_{MN})/2$, leading to

$$F_{MN}(\text{opt}) = \frac{M + N + MN}{N(M+2)}.$$
 (6.13)

This result had been obtained previously by Gisin and Massar [65] for $1\rightarrow N$ cloning. Although they proved it to be optimal for $N\leq 7$, they conjectured it to be optimal for all N. This conjecture was confirmed by Bruss, Ekert and Macchiavello [66]. These authors also showed that the optimum figures of merit for universal cloning are intimately related to the maximum fidelity obtained in universal state estimation. The relationship between universal state estimation and cloning is most easily expressed in terms of their respective shrinking factors S_M and S_{MN} . As is the case with unambiguous discrimination and exact cloning, the bounds on the optimal figures of merit for both operations imply each other.

C. Relationship between state estimation and universal cloning

In the remainder of this section, we will present the arguments of Bruss et al which show that, given the optimal shrinking factor $S_{MN}(\text{opt})$ for cloning, we can directly deduce the optimum shrinking factor, $S_M(\text{opt})$, for state estimation, and vice versa. To deduce the optimum shrinking factor for state estimation from that on cloning, suppose that an optimum state estimation measurement is carried out on M copies of an unknown state $|\psi\rangle$. We can use this state estimation procedure to make approximate copies of the state in the following way: every time the guess state $|\omega_k\rangle$ is obtained, we make N-M further copies of this guess state, for some N>M. The shrinking factor for the average guess state of each of these copies is identical to that for optimum state estimation. However, it cannot exceed that for an optimum UCM: it can at most equal it, in which case the procedure we describe would actually be an optimum UCM. This leads to the inequality

$$S_{MN}(\text{opt}) \ge S_M(\text{opt}),$$
 (6.14)

for all N>M. On the other hand, suppose that given M initial copies of the state $|\psi\rangle$, we send these states to an optimum $M{\to}N$ UCM. The N approximate copies of the state $|\psi\rangle$ are then subjected to an optimal state estimation measurement. The concatenation of an $M{\to}N$ UCM and a state estimation measurement on the N approximate copies cannot lead to a higher shrinking factor than a state estimation measurement on the M original copies (for much the same reason that a probabilistic cloning machine cannot be used to increase the probability of unambiguous discrimination, as we saw in the preceding section). Bruss et al showed that this leads to the inequality

$$S_{M\infty}(\text{opt}) \le S_M(\text{opt}).$$
 (6.15)

Combining inequalities (6.14) and (6.15), we see that the optimal shrinking factors for state estimation and infinite universal cloning are equal:

$$S_{M\infty}(\text{opt}) = S_M(\text{opt}). \tag{6.16}$$

The optimum shrinking factor, and thus fidelity, of state estimation can be deduced from that on universal cloning.

Let us now see how the optimal shrinking factor for cloning can be deduced from the optimum state estimation shrinking factor. This follows from a natural property of an optimal UCM which we will describe first. Suppose that we initially have M copies of the state $|\psi\rangle$. These are fed into an $M{\to}N$ UCM, and the corresponding shrinking factor is S_{MN} . We then feed these N approximate copies to a further $N{\to}L$ UCM, for some L>N. This will shrink the Bloch vector further, by the shrinking factor S_{NL} . Consider now an optimal $M{\to}L$ UCM. If this has the shrinking factor $S_{ML}(\text{opt})$, then clearly we must have

$$S_{MN}S_{NL} < S_{ML}(\text{opt}). \tag{6.17}$$

This holds for all L, in particular, in the limit as $L\rightarrow\infty$. It also holds whether or not the $N\rightarrow L$ UCM is optimum, so

$$S_{MN} \le \frac{S_{M\infty}(\text{opt})}{S_{N\infty}(\text{opt})}.$$
 (6.18)

To obtain an expression for the optimal shrinking factor $S_{MN}(\text{opt})$, we make use of two results. The first is the fact that, for concatenated optimal UCMs of the kind we have been describing, the shrinking factors multiply: that is, inequality (6.17) becomes an equality when $S_{MN} = S_{MN}(\text{opt})$. The second is Eq. (6.16), relating the shrinking factors for optimal infinite cloning and state estimation. These results imply that

$$S_{MN}(\text{opt}) = \frac{S_M(\text{opt})}{S_N(\text{opt})}.$$
 (6.19)

Notice the formal resemblance between this relationship and that between the maximum probabilities of unambiguous discrimination and exact cloning in Eq. (5.5). In fact, both arguments parallel one another, suggesting that a deeper and more general connection between determining the state of a quantum systems, and copying it, could be found.

We conclude this final section by mentioning some interesting subsequent developments related to universal state estimation and cloning. The results we have discussed in this section refer to an unknown state of a 2-level system. It is natural to enquire as to how these results can be generalised to the case of multilevel systems. The generalisation of universal cloning machines to multilevel systems has been fully worked out by Werner and Keyl [67,68], who gave an elegant mathematical characterisation of the optimal such transformations.

The relationship between universal cloning and state estimation for multilevel quantum systems has also been explored by Bruss and Macchiavello [69]. For general systems, this relationship is exactly as was shown for the 2 level case, and the same arguments can be used to deduce the optimal figures of merit.

Although we have examined the 2-level case in some detail, the results we have described by no means tell the whole story, even about this simple case. We were interested in estimating, or copying, a state $|\psi\rangle$, given M initial copies. Suppose that the physical systems which were prepared in this state were spin-1/2 particles. The classical analogue of such a system would be M particles whose spins point in the same direction. We would thus expect that, if some of the spins were anti-parallel to the others, the information content would be the same, since they define the same spin axis. Gisin and Popescu [70] investigated this from a quantum mechanical point of view, and found that this is not the case, namely, that anti-parallel spins contain more information.

One of the most exciting recent developments in the field has been the announcement, just a few days before completion of this article, that universal cloning of photon polarisation has been carried out in the laboratory by Li et al [71]. This experiment, together with contemporaneous demonstrations of optimal quantum state discrimination [21,28], provide further encouragement to explore the ultimate physical limits of information processing and transmission.

VII. DISCUSSION

Although the state of a quantum system is not itself an observable, we have seen how novel measurement strategies, which are completely consistent with the formalism of quantum mechanics, enable one to obtain information about it.

Often these measurements are different from the standard von Neumann type discussed in introductory quantum mechanics courses. They may instead be generalised measurements. These typically involve the interaction of the system with another ancillary system, after which a

von Neumann measurement is carried out on the latter. These more general measurement strategies offer greater flexibility and scope than operations performed upon the system of interest alone. One of the main advantages conferred by generalised measurements to the study of state discrimination is the fact that the number of outcomes is not limited to the dimension of the system's state space. In quantum hypothesis testing, this allows us to discriminate between an arbitrarily large number of states with some probability of obtaining a correct result. An additional outcome can correspond to inconclusive results, which allows linearly independent states to be unambiguously discriminated.

We also saw how state discrimination is related to other operations on quantum systems, such as cloning and the manipulation of quantum entanglement. The field of quantum information has enjoyed rapid growth over the past few years, and some of the most intriguing discoveries made about the information-theoretic properties of quantum systems have been unifications of seemingly distinct concepts such as those we have discussed here.

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